Generalized q-dimension of measures on Heisenberg self-affine sets in Heisenberg group

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Heisenberg group

The *Heisenberg group* $\mathbb{H}$ is the unique analytic, nilpotent Lie group of dimension 3 whose background manifold is $\mathbb{R}^3$ and whose Lie algebra $\mathfrak{h}$ satisfies the following properties:

- $\mathfrak{h} = V_1 \oplus V_2$ where V_1 has dimension 2 andV_2 has dimension 1.
- $[V_1, V_1] = V_2$, $[V_1, V_2] = 0$ and $[V_2, V_2] = 0$.

Explicitly, let $x = (x_1, x_2) \in \mathbb{R}^2$, and the mapping $J : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$J(x_1, x_2) = (-x_2, x_1).$$

The group law on $\mathbb{H} \cong \mathbb{R}^3$ is defined by

$$\mathbf{x} * \mathbf{y} = (x + y, t + s + 2\langle x, Jy \rangle), \qquad (1)$$

for all $\mathbf{x} = (x, t)$, $\mathbf{y} = (y, s) \in \mathbb{H}$, where $x, y \in \mathbb{R}^2$, $t, s \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^2$.

Let d_H be the *Heisenberg metric* given by

$$d_H(\mathbf{x},\mathbf{y}) = \|\mathbf{x}^{-1} * \mathbf{y}\|_H, \quad \text{for} \quad \mathbf{x},\mathbf{y} \in \mathbb{H},$$

where $*$ denotes the group law defined in (1), and $\|\cdot\|_H$ denotes the Heisenberg norm given by

$$\|(x,t)\|_H = (|x|^4 + t^2)^{1/4}.$$

Note that the Heisenberg group $(\mathbb{H}, d_H)$ equipped with the non-Euclidean metric is a complete metric space. Note that the Heisenberg metric d_H is left-invariant on $\mathbb{H}$.

horizontal lift

Given a mapping $s : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\pi : \mathbb{H} \to \mathbb{R}^2$ be given by $\pi(\mathbf{x}) = x$, where $\mathbf{x} = (x, t) \in \mathbb{H}$.
We call $S : \mathbb{H} \to \mathbb{H}$ a *horizontal lift* of s if

$$\pi \circ S = s \circ \pi.$$

Let $s : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformation, ie.

$$s(x) = Tx + a,$$

where T is a real 2×2 matrix and a is a vector in $\mathbb{R}^2$.
Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be an affine mapping of the form

$$S(x,t) = \begin{pmatrix} T & b \\ d & c \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} a \\ \eta \end{pmatrix}, \qquad (2)$$

where $a, b, d \in \mathbb{R}^2$ and $c, \eta \in \mathbb{R}$.

The horizontal lift S is *Lipschitz* with respect to the metric d_H if and only if we have that

$$b = 0, \quad d = -2T'Ja \text{ and } \qquad c = \det T.$$

Thus, we write a Lipschitz affine map $S : \mathbb{H} \to \mathbb{H}$ as

$$S(\mathbf{x}) = \widetilde{T}_a \mathbf{x} + \widetilde{a}, \tag{3}$$

where for some real constant η,

$$\widetilde{T}_a = \begin{pmatrix} T & 0 \\ -2(Ja)'T & \det T \end{pmatrix}, \quad \widetilde{a} = \begin{pmatrix} a \\ \eta \end{pmatrix}. \tag{4}$$

Iterated function system in Heisenberg group

Given a self-affine IFS $\{s_i(x) = T_i x + a_i\}_{i=1}^M$. We write $\mathcal{F}(\mathbf{a})$ for the attractor of $\{s_i(x)\}_{i=1}^M$, that is,

$$\mathcal{F}(\mathbf{a}) = \bigcup_{i=1}^M S_i(\mathcal{F}(\mathbf{a})). \tag{5}$$

Applying the horizontal lift onto $\{s_i(x)\}_{i=1}^M$, we obtain a class of contractions $\{S_1, \cdots, S_M\}$, called a *self-affine IFS on Heisenberg group*, where the mappings S_i are given by

$$S_i(\mathbf{x}) = \widetilde{T}_{i,a_i}\mathbf{x} + \widetilde{a}_i, \qquad i = 1, \ldots, M, \tag{6}$$

and $\widetilde{T}_{i,a_i}$ and $\widetilde{a}_i$ are as in (4). We write $\widetilde{\mathbf{a}} = (\widetilde{a}_1, \cdots, \widetilde{a}_M) \in \mathbb{R}^{3M}$. Similarly, We call $\mathcal{F}_H(\widetilde{\mathbf{a}})$ the *Heisenberg self-affine set* for $\{S_i(\mathbf{x})\}_{i=1}^M$,

$$\mathcal{F}_H(\widetilde{\mathbf{a}}) = \bigcup_{i=1}^M S_i(\mathcal{F}_H(\widetilde{\mathbf{a}})). \tag{7}$$

Sequence space

For each $k = 0, 1, 2, \cdots$, let $\mathbf{J}_k = \{(i_1, \cdots, i_k) : 1 \le i_j \le M\}$ be the set of sequences of length k.
Let $\mathbf{J} = \bigcup_{k=0}^{\infty} \mathbf{J}_k$ be the set of all finite sequences.
Let $\mathbf{J}_\infty = \{(i_1, i_2, \cdots) : 1 \le i_j \le N\}$ be the corresponding set of infinite sequences.
We abbreviate members of $\mathbf{J}$ or $\mathbf{J}_\infty$ as $\mathbf{i} = (i_1, \cdots, i_k)$, etc., and denote the number of terms in each word $\mathbf{i} \in \mathbf{J}$ by $|\mathbf{i}|$.
If $\mathbf{i}, \mathbf{j} \in \mathbf{J}$ or if $\mathbf{i} \in \mathbf{J}$ and $\mathbf{j} \in \mathbf{J}_\infty$, we denote by $\mathbf{ij}$ the sequence obtained by concatenation of the words $\mathbf{i}$ and $\mathbf{j}$.
We write $\mathbf{j}|_k \in \mathbf{J}_k$ for the initial k-term sequence of $\mathbf{j} \in \mathbf{J}_\infty$ and define the cylinders $\mathcal{C}_{\mathbf{i}} = \{\mathbf{j} \in \mathbf{J}_\infty : \mathbf{j}|_k = \mathbf{i}\}$.
Let $\mathbf{I}$ be a finite subset of $\mathbf{J}$, we say $\mathbf{I}$ is a *cut set* of $\mathbf{J}$ if for every $\mathbf{i} \in \mathbf{J}_\infty$ there is a unique integer k such that $\mathbf{i}|k \in \mathbf{I}$. For such a cut-set $\mathbf{I}$ we write $k(\mathbf{I}) = \min\{|\mathbf{i}| : \mathbf{i} \in \mathbf{I}\}$.

singular-value functions

Let T be a nonsingular real 2×2 matrix. The *singular values* α_1, α_2 are the positive square roots of the eigenvalues of TT', where T' is the transpose of T.

We adopt the convention that $1 > \alpha_1 \geq \alpha_2 > 0$.

The *Heisenberg singular value function* $\psi^s(T)$ in the Heisenberg group is defined to be

$$\psi^s(T) = \begin{cases} \alpha_1^s & \text{if } 0 \leq s \leq 1, \\ \alpha_1^{(s+1)/2}\alpha_2^{(s-1)/2} & \text{if } 1 < s \leq 3, \\ \alpha_1^2\alpha_2^{s-2} & \text{if } 3 < s \leq 4, \end{cases} \tag{8}$$

with the convention that $\psi_s(T) = \alpha_1^{s/2}\alpha_2^{s/2}$ if $s \geq 4$.

Generalized q-dimension in Heisenberg group

Given $0 < r < 1$, let

$$\mathcal{M}_r = \{C_H(\gamma, r) : \gamma \in \Gamma_r\}, \quad \Gamma_r = \left\{(2ru, v) \in \mathbb{H} : u \in \mathbb{Z}^2, \frac{v}{2r^2} \in \mathbb{Z}\right\}.$$

Let τ be a finite Borel measure with bounded support denoted by $spt\tau < \infty$. Then, for $q \in \mathbb{R}$, the *moment sum* $M_r(q)$ is defined to be

$$M_r(q) = \sum_{C \in \mathcal{M}_r} \tau(C)^q,$$

where the sum is over all $r-$mesh cubes C such that $C \bigcap spt\tau \neq \emptyset$.
For $q \neq 1$, we define the lower and upper generalized q-dimensions of τ by setting

$$\underline{D}_q^H(\tau) = \liminf_{r \to 0} \frac{\log M_r(q)}{(q-1)\log r}, \qquad \overline{D}_q^H(\tau) = \limsup_{r \to 0} \frac{\log M_r(q)}{(q-1)\log r}.$$

If the quantities $\underline{D}_q^H$ and $\overline{D}_q^H$ are identical, we write it as D_q^H
the quantities $\underline{D}_q^H$ and $\overline{D}_q^H$ can also be defined in terms of integrals when $q \neq 1$.

$$\underline{D}_q^H(\tau) = \liminf_{r \to 0} \frac{\log M_r(q)}{(q-1)\log r} = \liminf_{r \to 0} \frac{\log \int \tau(B_H(\mathbf{x}, r))^{q-1} \mathrm{d}\tau(\mathbf{x})}{(q-1)\log r},$$

$$\overline{D}_q^H(\tau) = \limsup_{r \to 0} \frac{\log M_r(q)}{(q-1)\log r} = \limsup_{r \to 0} \frac{\log \int \tau(B_H(\mathbf{x}, r))^{q-1} \mathrm{d}\tau(\mathbf{x})}{(q-1)\log r}.$$

Self-affine measure on Heisenberg group

We may define a Borel measure μ on $\mathbf{J}_\infty$ by setting

$$\mu(\mathcal{C}_{\mathbf{i}}) = p_{i_1} p_{i_2} \cdots p_{i_k}, \tag{9}$$

with probability vector $(p_1, \cdots, p_M)$ on the cylinders $C_{\mathbf{i}}$.
Let $\mu^{\widetilde{\mathbf{a}}}$ on $\mathbb{H}$ be a self-affine measure, ie.

$$\mu^{\widetilde{\mathbf{a}}}(B) = \sum_{i=1}^{M} p_i \mu^{\widetilde{\mathbf{a}}}(F_i^{-1}(B)).$$

For all $q \geq 0$, $q \neq 1$ there exists a unique positive number $l_q \equiv l_q(T_1, \cdots, T_M; \mu)$ such that

$$\lim_{k\to\infty} \Big(\sum_{\mathbf{i}\in\mathbf{J}_k} \psi^{l_q}(T_{\mathbf{i}})^{1-q} \mu(\mathcal{C}_{\mathbf{i}})^q \Big)^{\frac{1}{k}} = 1. \tag{10}$$

Generalised q-dimension of Self-affine measure on Heisenberg group

Theorem (with Wu)

Let $(S_1, \cdots, S_M)$ *be an affine Heisenberg IFS, let* $(p_1, \cdots, p_M)$ *be probabilities with* μ *the measure on* $\mathbf{J}_\infty$ *defined by* (9) *and let* $l_q = l_q(S_1, \cdots, S_M; \mu)$ *be given by* (10)*. For each* $\widetilde{\mathbf{a}} \in \mathbb{R}^{3M}$ *we define* $\mu^{\widetilde{\mathbf{a}}}$ *to be a self-affine measure.*

(a) *If* $q \geq 0$, $q \neq 1$, *then* $\overline{D}_q^H(\mu^{\widetilde{\mathbf{a}}}) \leq \min\{l_q, 4\}$ *for all* $\widetilde{\mathbf{a}} \in \mathbb{R}^{3M}$.

(b) *If* $1 < q \leq 2$ *and* $\|S_j\| < \frac{1}{2}$ *for all* $j = 1, \cdots, M$, *then* $D_q^H(\mu^{\widetilde{\mathbf{a}}}) = \min\{l_q, 4\}$ *for* $\mathcal{L}^{3M}$*-almost all* $\widetilde{\mathbf{a}} \in \mathbb{R}^{3M}$.