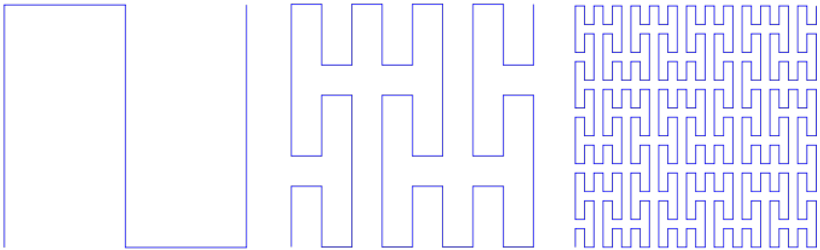
Our Goals

Describe a new method, building upon previous research on Julia sets, to contruct Laplacians on a self-similar set X using a Peano curve from the circle onto X.

Unique approach because in our method the Peano curves exhibit self-intersections which will play a vital role in constructing the graph approximations to the fractal.



Introduction

Peano Curve: If X is a compact topological space, we use the term *Peano curve* for any continuous mapping γ from the unit circle (parameterized by $t \in [0, 1]$ such that $0 \equiv 1$) onto X.

Key Points:

Such a mapping, γ, can never be one-to-one, thus there must exist points $t_1, t_2 \in [0, 1]$ such that $\gamma(t_1) = \gamma(t_2) \in X$. We will say that $t_1 \equiv t_2$ and call these points "*identified points*"

If we consider all possible identifications we can obtain a model of X as a circle with appropriately identified points.

Introduction

Graph Laplacian: Given a finite set of identifications on the circle, we have a natural graph structure where the points are the vertices, and the edges join the consecutive points around the circle. We assign positive weights $\mu(t_j)$ to the points and think of these as a discrete measure on the set of vertices. We assign non-negative weight $c(t_j, t_{j+1})$ to the edges.

Discrete Laplacian:

$$-\triangle u(x) = \frac{1}{\mu(x)} \sum c(x, y)(u(x) - u(y)) \tag{0.1}$$

The Self-Similar sets X(Fractal Examples)

The Pentagasket (PG)

Attributes: Postcritically-finite (PCF) self-similar set, fully symmetric self-similar Laplacian with well-known properties, does not satisfy spectral-decimation.

The Octagasket (OG)

Attributes: Non-PCF fractal, only experimental evidence of self-similar Laplacian, interesting locations in the spectrum for spectral gaps.

The Magic Carpet (MC)

Attributes: Non-PCF fractal, constucted by modifying the Sierpinksi carpet, only eperimental evidence for self-similar Laplacian.

The Self-Similar sets X(Non-Fractal Examples)

Equilateral Triangle(T)

Attributes: Non-fractal, Neumann boundary conditions, acts as a control as widely studied spectrum

The Square Torus (T_0)

Attributes: Non-fractal, acts as a control

Constructing the Peano Curves

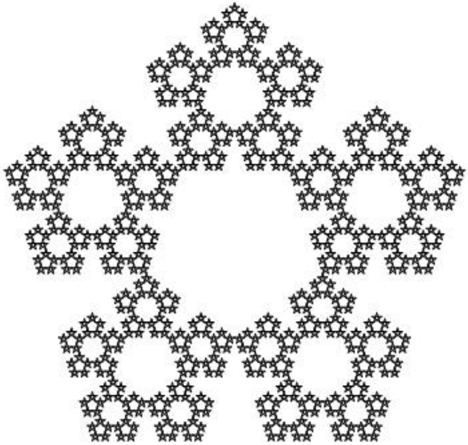
Main Ideas:

The Peano curves are all constructed using piecewise linear maps, γ_m, where the passage from γ_m to γ_{m+1} is given by a set of substitution rules.

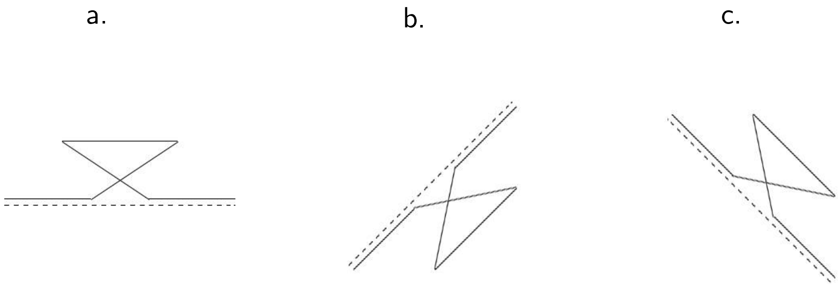
The line segments will be the edges of X and the end points of γ_m will give forth the vertices of X. These edges and vertices are then associated with points and edges on the circle.

When passing to the limit γ_m will give rise to X.

The Pentagasket

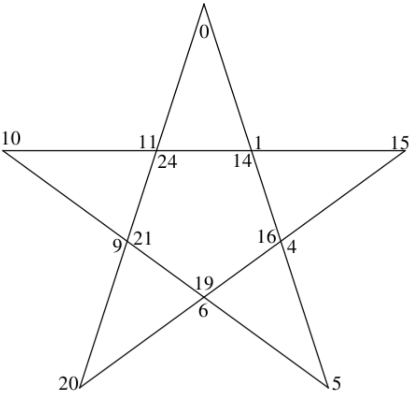


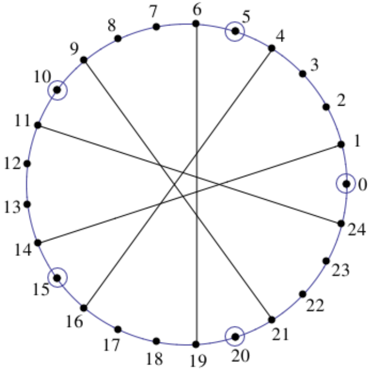
Substitution rule for PG



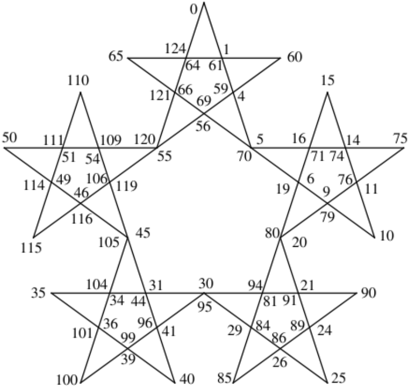
The dotted line shows the replaced line segment from the previous level of the graph approximation. The other substitutions at the same level are obtained by rotating through angles of $\frac{2\pi k}{5}$.

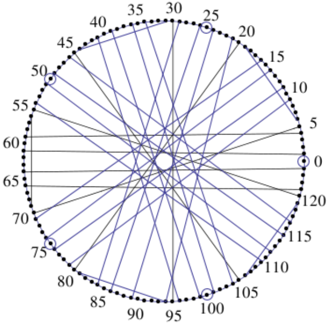
Identifications at Level 1



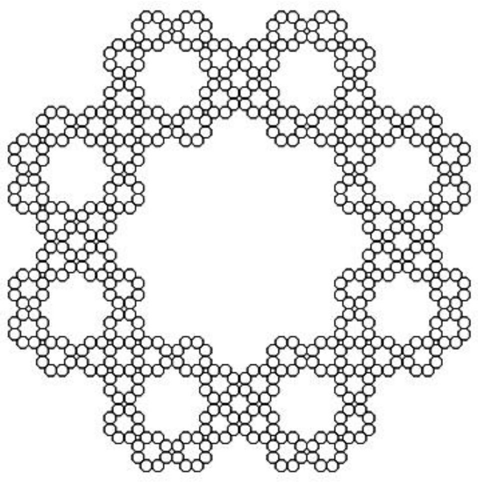


Identifications at Level 2

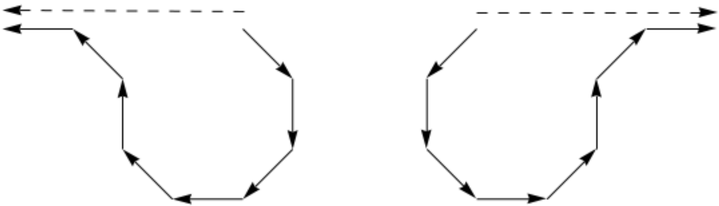




The Octagasket

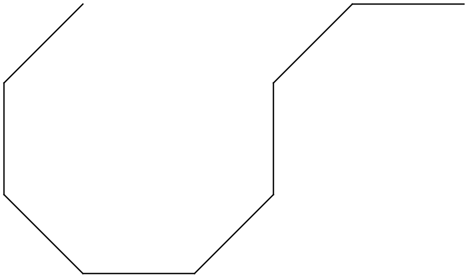


Substitution Rule for OG



The dotted arrow shows the line from the previous level of the graph approximation that is to be replaced. Note that the two substitutions differ in direction and are reflections of one another. There are eight total substitutions which are just rotations of these. Each rotation corresponds to a different possible edge of the OG graph approximation.

Octagasket



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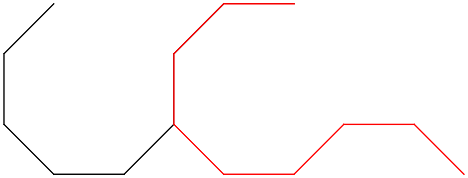

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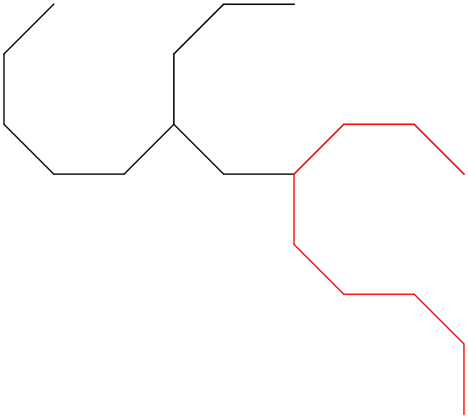

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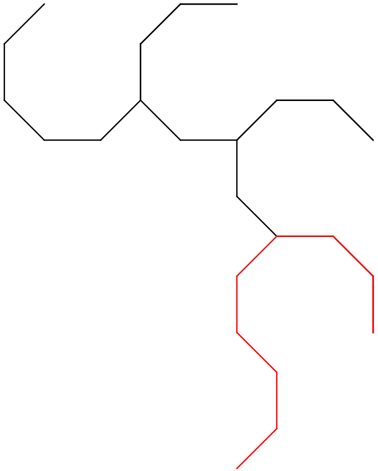

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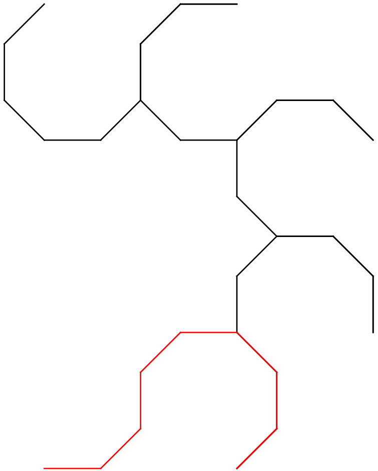

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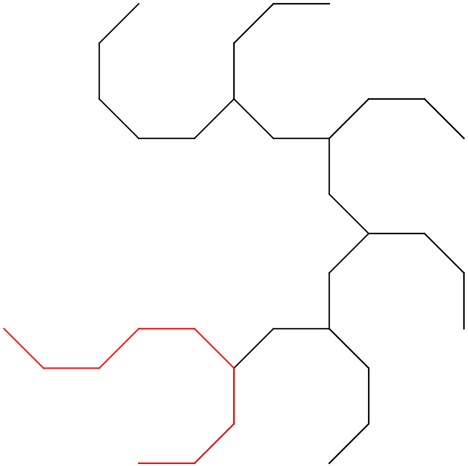

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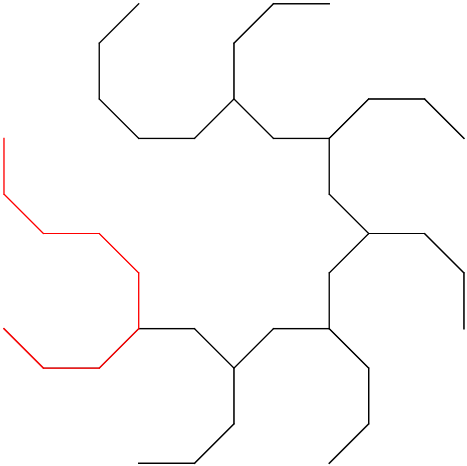

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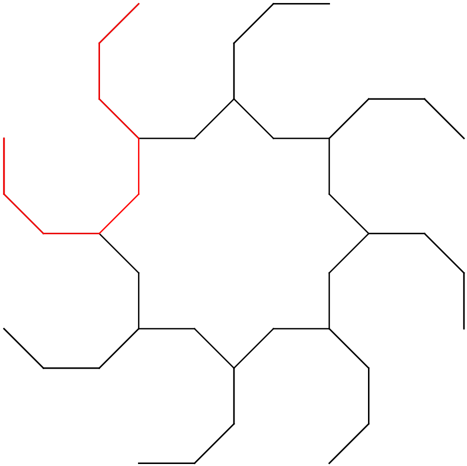

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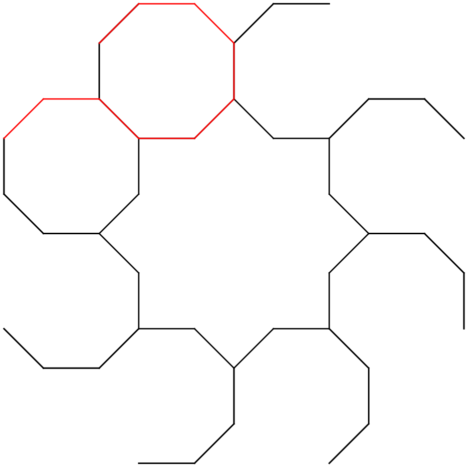

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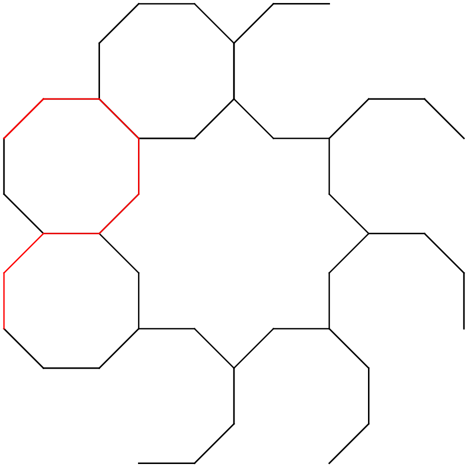

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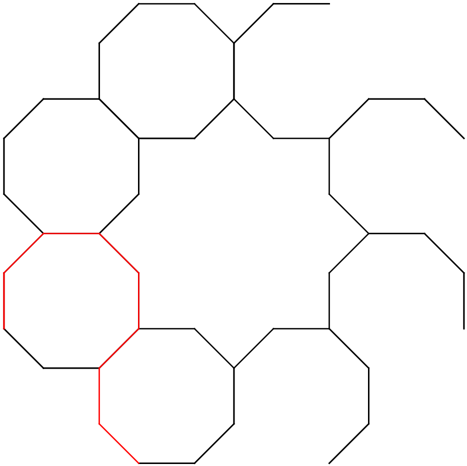

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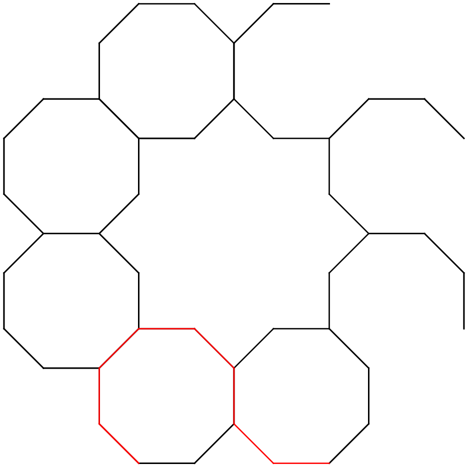

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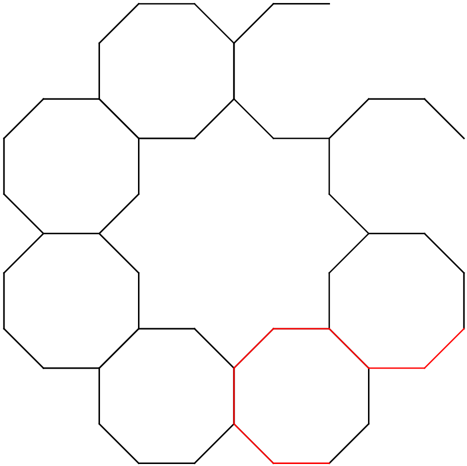

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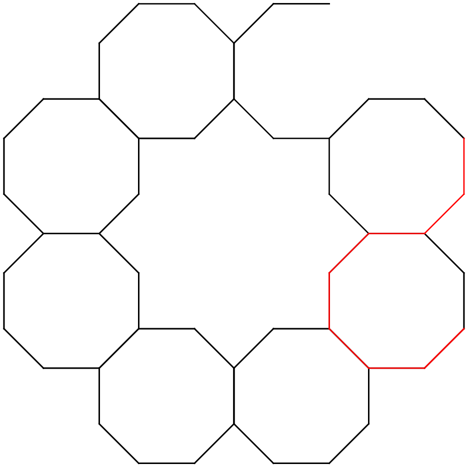

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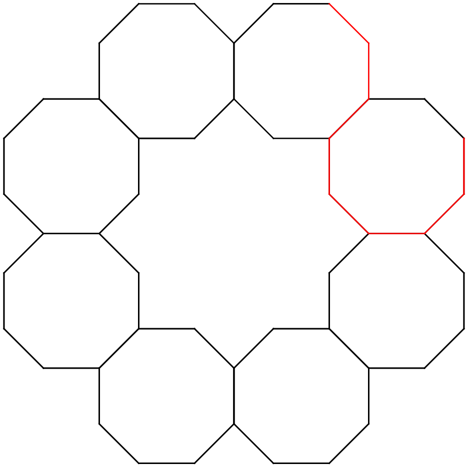

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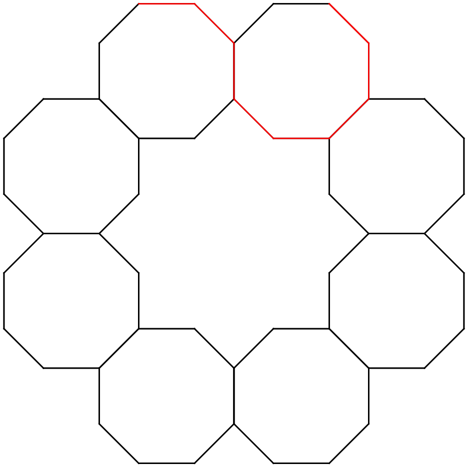
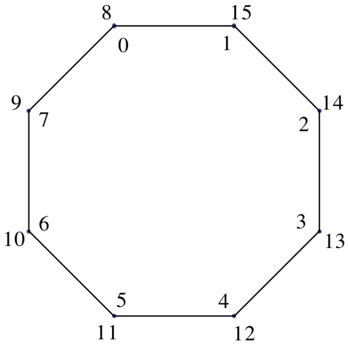
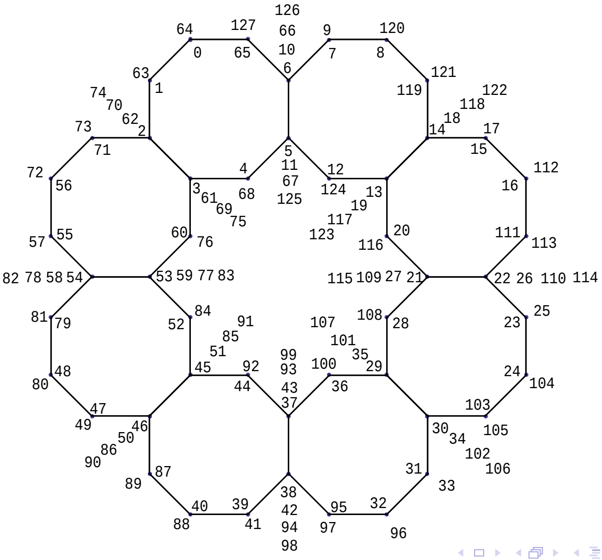


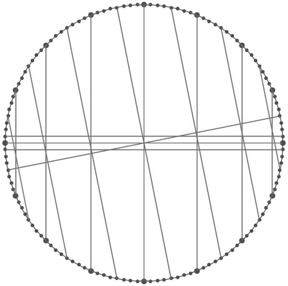
The path of γ_0



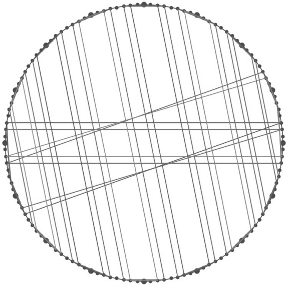
The path of γ_1



Intersections of γ on OG



a.

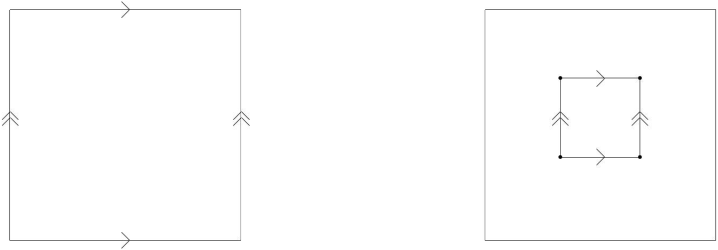


b.

In the case of OG, γ intersects in sets of two and four. We call the equivalence class consisting of two points *outer points* (a.) while the equivalence classes consisting of four points are called *inner points* (b.)

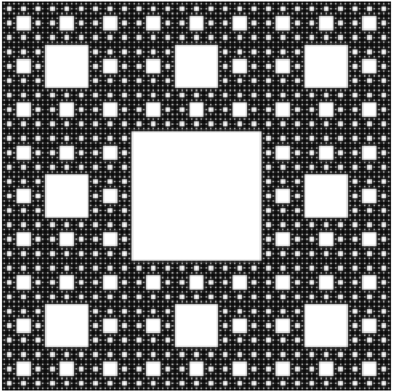
The Magic Carpet

The Magic Carpet is obtained by modifying the construction of the Sierpinksi Carpet by immediately sewing up all cuts that are made.

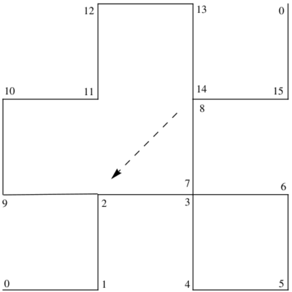


a. b.

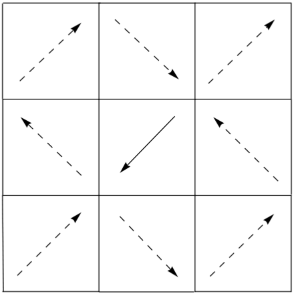
Sierpinksi Carpet Reference



The path γ_1 for MC



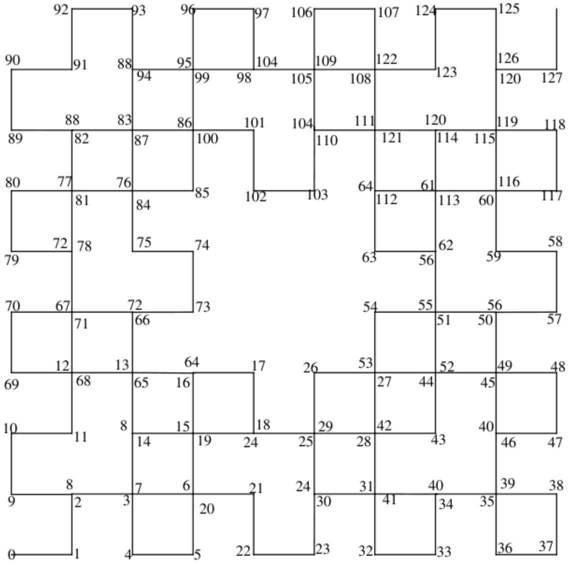
a.



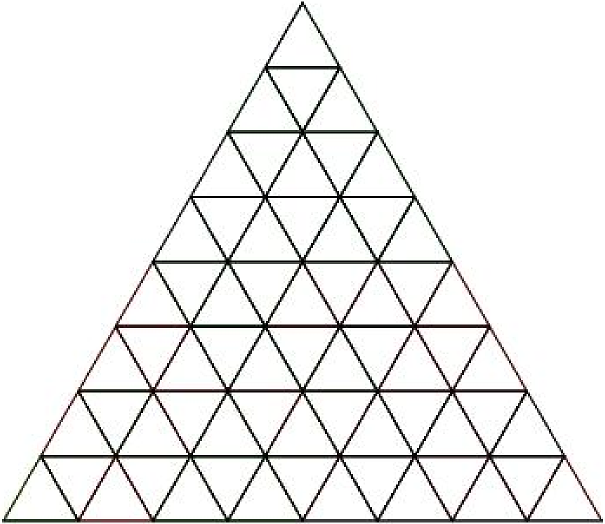
b.

The path γ_1 for MC. In a. we show the actual path with the points 2,3,7,8,11 and 14 identified. In b., the dashed arrows show the symbolic description, while the solid arrow illustrates a jump between identified points.

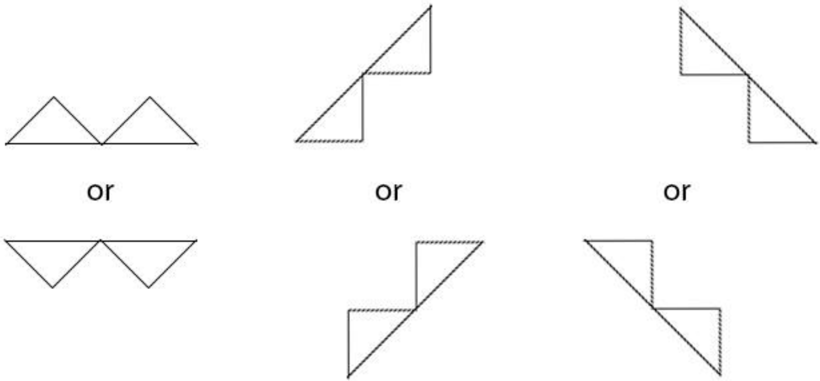
The path of γ_2 on MC



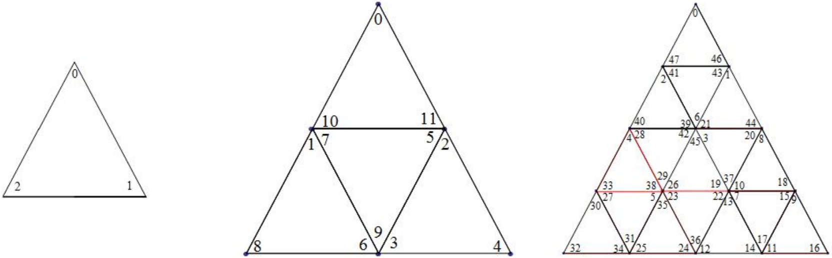
The Equilateral Triangle(T)



Substitution Rule for the Triangle



The path of γ_0,γ_1 and γ_2 on T



The interesections of γ

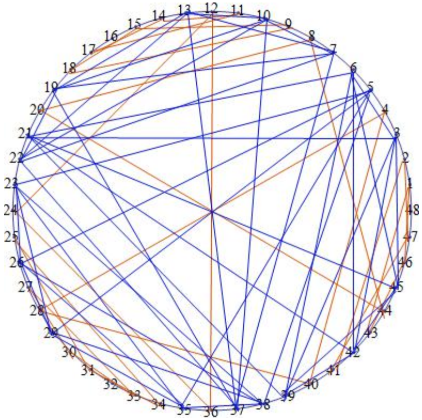
Characterization of points:

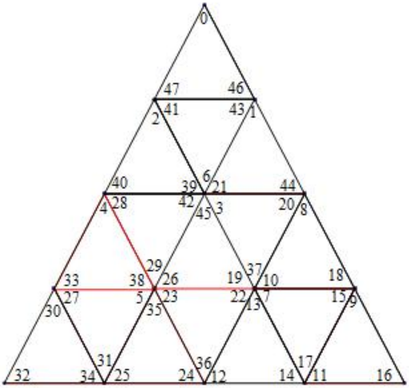
Corner Points: No identifications

Outer Points: The other points along the boundary are identified in groups of three

Interior Points: Identified in groups of six

Identifications of γ_2





Graph Energies

We define graph energy as

$$E_G(u) = \sum_{x \sim y} (u(x) - u(y))^2 \tag{0.2}$$

where u is a continuous function on the circle that respects all identifications made by γ.

Let $V \subseteq V'$ such that $u\big|_V$. We want to extend our function $u\big|_V$ to $u'\big|_{V'}$. We aim to find $\tilde{u}$ such that $\tilde{u}$ minimizes $E_{V'}(\tilde{u})$. We call such a function $\tilde{u}$ a *harmonic extension*.

Renormalization factor

$$E_{\gamma_1}(u) = rE_{\gamma_0}(u) \tag{0.3}$$

where $0 < r < 1$.

We call r the *renormalization factor*.

Renormalized Energy:

$$\mathcal{E}(u) = r^{-m}E(u) \tag{0.4}$$

Self-similar measure μ

We call μ a *regular probability measure*.

μ assigns weights to the vertices of X. In our case, μ depends on the size of the equivalence classes of the vertices.

PG Conductance $c(x, y)$ and renormalization factor r

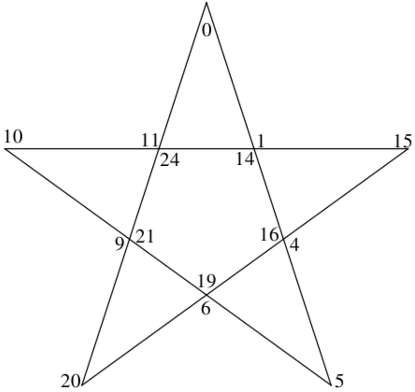
To calculate $\triangle u(x)$ we must first designate a conductance along the edges of PG as well as calculate r.

$$c(x,y) = \begin{cases} 1 \text{ if } [x,y] \text{ has length } \frac{1}{5^{m+1}} \\ b \text{ if } [x,y] \text{ has length } \frac{3}{5^{m+1}} \end{cases}, \tag{0.5}$$

where $b = \frac{1+\sqrt{161}}{10}$ was determined using basic principles of electric network theory, applied to pieces of the graphs.

Using aforementioned process, we found $r = \frac{\sqrt{161}-9}{8}$

PG Reference



PG Laplacian $\triangle u_m(x)$

Weights assigned:

$$\mu(x) = \begin{cases} \frac{1}{5^{m+1}} & \text{if } k \equiv 0 \mod 5 \\ \frac{2}{5^{m+1}} & \text{if } k \equiv 1 \text{ or } 4 \mod 5 \end{cases} \tag{0.6}$$

Where $\mu(x)$ is the sum of the weights of all points in the equivalence class.

PG Eigenvalues

Level 1			Level 2			Level 3			Level 4		
#	Mult	Eigenvalue	#	Mult	Eigenvalue	#	Mult	Eigenvalue	#	Mult	Eigenvalue
1	1	0	1	1	0	1	1	0	1	1	0
2	2	28.6410	2	2	12.5186	2	2	12.6700	2	2	12.6832
4	2	28.9251	4	2	30.6109	4	2	31.3706	4	2	31.4492
6	2	119.5409	6	5	143.2049	6	5	135.7523	6	5	137.4025
8	2	132.5555	11	1	168.8936	11	1	164.5714	11	1	166.9378
10	1	135.5536	12	2	182.4264	12	2	182.3916	12	2	185.2678
			14	2	215.2990	14	2	239.2249	14	2	244.1480
			16	5	415.7326	16	5	331.9515	16	5	340.1929
			21	2	430.6319	21	2	435.5986	21	2	453.4902
			23	2	454.5580	23	2	562.4423	23	2	596.8892
			25	1	463.5525	25	1	629.634	25	1	677.4916
			26	5	597.7066	26	20	1552.9561	26	20	1472.1417

Eigenvalue Counting Function and Weyl Ratio

Eigenvalue Counting Function: $\rho(x) = \#\{\lambda_j \leq x\}$

Weyl Ratio: $WR(x) = \frac{\rho(x)}{x^{\beta}}$ where β is the slope of the best line of $\rho(x)$.

β values:

PG($\beta \approx 0.675$)

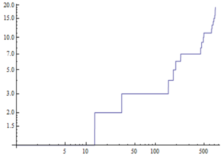
OG($\beta \approx 0.7213$)

MC ($\beta \approx 1.2$)

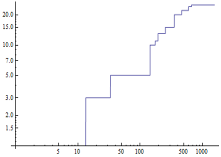
Eigenvalue Counting Function and Weyl Ratio

Eigenvalue Counting Function

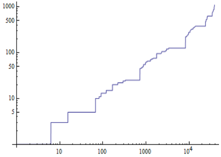
Level 2



Level 3

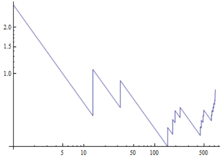


Level 4

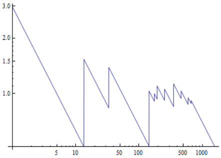


Weyl Ratios

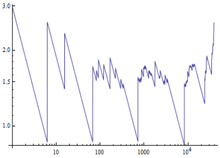
Level 2



Level 3

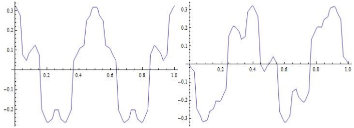


Level 4

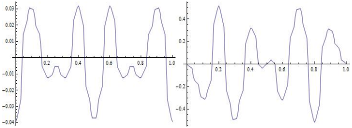


PG Eigenfunctions

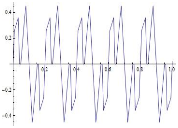
Eigenfunction # 2 & 3



Eigenfunction # 4 & 5

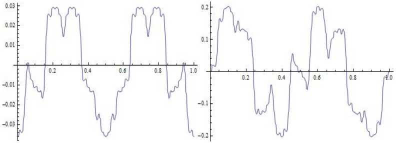


Eigenfunction # 11

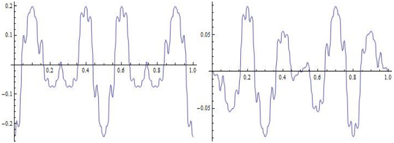


PG Eigenfunctions

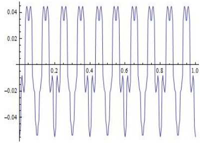
Eigenfunction # 2 & 3



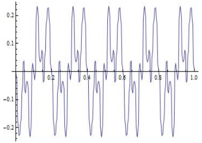
Eigenfunction # 4 & 5



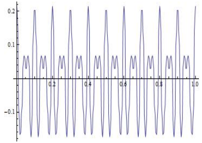
Eigenfunction #6



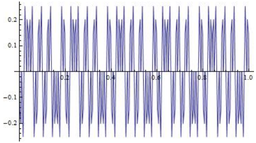
Eigenfunction #11



Eigenfunction # 16



Eigenfunction # 51



OG $\triangle u(x)$

Assigned weights:

$$\mu(x) = \begin{cases} \frac{1}{8^{m+1}} & \text{if } \textit{outer point} \\ \frac{2}{8^{m+1}} & \text{if } \textit{inner point} \end{cases} \tag{0.7}$$

$$-\Delta_m u(x) = 4(u(x) - \text{Ave}(u(y))) \tag{0.8}$$

The ratios of corresponding eigenvalues let us estimate $r \approx 0.537$.

And so we have,

$$-\Delta u = \lim_{m \to \infty} (\frac{8}{r})^m \triangle_m u \tag{0.9}$$

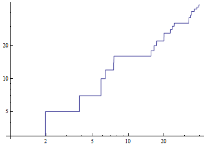
Eigenvalues of the Octagasket

Level 1			Level 2			Level 3			Ratio	
#	Mult	Eigenvalue	#	Mult	Eigenvalue	#	Mult	Eigenvalue	$\frac{\lambda_1}{\lambda_2}$	$\frac{\lambda_2}{\lambda_3}$
1	1	0	1	1	0	1	1	0		
2	2	0.111	2	2	0.0074	2	2	0.0005	14.802	14.938
4	2	0.396	4	2	0.0282	4	2	0.0018	14.027	14.908
6	2	0.770	6	2	0.0570	6	2	0.0038	13.495	14.897
8	3	1.171	8	1	0.0784	8	1	0.0052		14.960
11	2	1.276	9	2	0.1108	9	2	0.0074		14.794
13	2	1.500	11	2	0.1157	11	2	0.0077		14.852
15	2	1.506	13	2	0.1251	13	2	0.0083		14.941
17	2	3.109	15	2	0.1263	15	2	0.0084		14.971
19	2	3.299	17	2	0.2291	17	2	0.0154		14.803
21	2	3.465	19	1	0.2362	19	1	0.0157		15.034
23	4	4.000	20	2	0.2412	20	2	0.0165		14.590
27	2	4.534	22	2	0.2771	22	2	0.0189		14.605
29	2	4.700	24	1	0.3021	24	1	0.0205		14.691
31	2	4.890	25	2	0.3961	25	2	0.0282		14.027
33	2	6.493	27	2	0.4237	27	2	0.0300		14.120
35	2	6.499	29	2	0.4261	29	2	0.0301		14.136

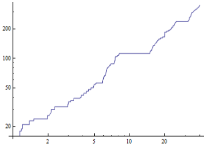
OG Eigenvalue Counting Function and Weyl Ratio

Eigenvalue Counting Functions

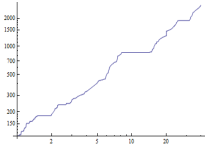
Level 1



Level 2

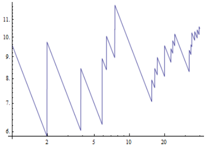


Level 3

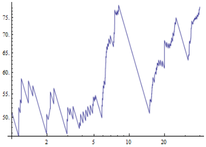


Weyl Ratios

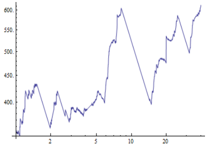
Level 1



Level 2



Level 3

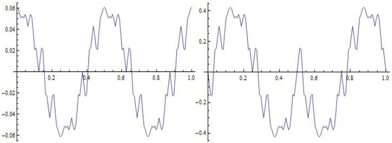


Spectral Gaps of the Octagasket

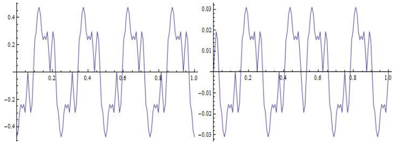
k	$16k$	$\frac{\lambda_{16k+1}}{\lambda_{16k}}$
1	16	1.8350
2	32	1.3236
7	112	1.554
15	240	1.168
54	864	1.768

OG Eigenfunctions

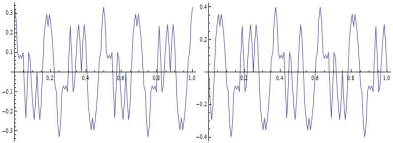
Eigenfunction # 2 & 3



Eigenfunction # 4 & 5

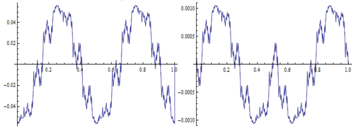


Eigenfunction # 6 & 7

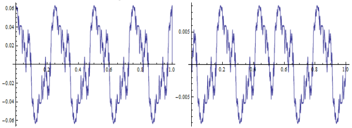


OG Eigenfunctions

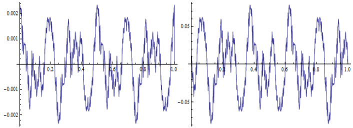
Eigenfunction # 2 & 3



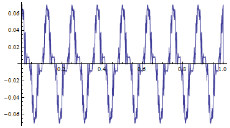
Eigenfunction # 4 & 5



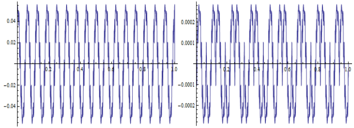
Eigenfunction # 6 & 7



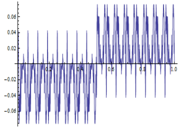
Eigenfunction # 8



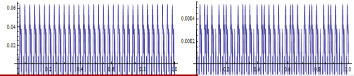
Eigenfunction # 9 & 10



Eigenfunction # 24



Eigenfunction # 25 & 26

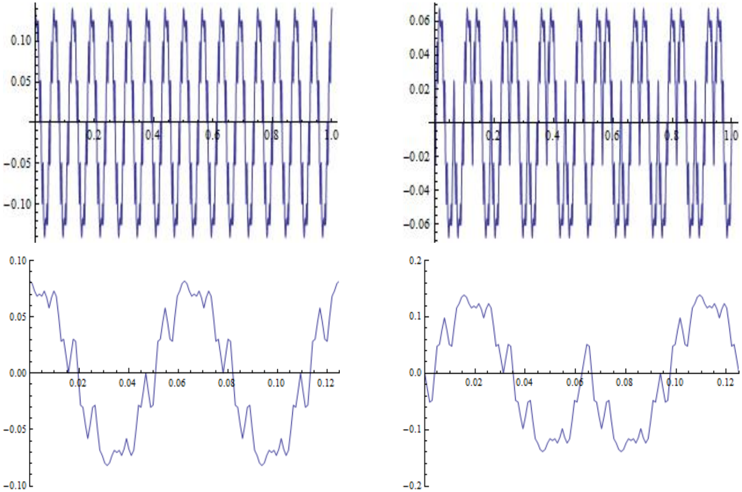


OG Miniaturization

Miniaturization of Eigenfunctions: Each eigenvalue of $-\Delta_m$ is also an eigenvalue of $-\Delta_{m+1}$ with the same multiplicity. The corresponding eigenfunction of $-\Delta_m$ is *miniaturized* to create the eigenfunctions of $-\Delta_{m+1}$.

The rule for miniaturization depends of the dihedral-8 symmetry group of the corresponding eigenspace.

OG Miniaturization



MC $\triangle u(x)$

Analagous to OG, we have

$$-\triangle_m u(x) = 12(u(x) - \text{Ave}(u(y))) \tag{0.10}$$

Some points, x, will have neighbors that are identified with it thus we have,

$$-\Delta_m(x) = \begin{cases} 12u(x) - 3\sum_{y\sim x} u(y) & \text{if } x \text{ has 2 identifications} \\ 12u(x) - \sum_{y\sim x} u(y) & \text{if } x \text{ has 6 identifications and 12 distinct neighbors} \\ 8u(x) - \sum_{y\sim x} u(y) & \text{if } x \text{ has 6 identifications and 8 distinct neighbors} \end{cases} \tag{0.11}$$

We also have $r \approx 1.25$. This is interesting because it means that r^m will blow up.

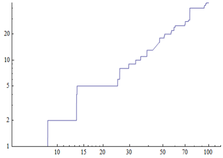
MC Eigenvalues

Level 1			Level 2			Level 3			Level 4			Ratio	
#	Mult	Eiv	#	Mult	Eiv	#	Mult	Eiv	#	Mult	Eiv	$\frac{\lambda_2}{\lambda_3}$	$\frac{\lambda_3}{\lambda_4}$
1	1	0	1	1	0	1	1	0	1	1	0		
2	1	9.000	2	1	1.726	2	1	0.274	2	1	0.0429	6.281	6.406
3	1	10.228	3	2	2.674	3	2	0.441	3	2	0.068	6.069	6.442
4	2	15.000	5	1	2.697	5	1	0.458	5	1	0.072	5.885	6.365
6	1	18.772	6	1	5.000	6	1	0.869	6	1	0.138	5.752	6.294
			7	2	5.515	7	2	0.923	7	2	0.146	5.586	6.296
			9	1	5.917	9	1	0.987	9	1	0.154	5.993	6.402
			10	1	6.580	10	1	1.112	10	1	0.173	5.915	6.423
			11	1	7.102	11	1	1.304	11	1	0.207	5.444	6.290
			12	2	7.808	12	2	1.431	12	2	0.223	5.455	6.412
			14	3	9.000	14	2	1.610	14	2	0.232		6.386
			17	2	9.475	16	1	1.620	16	1	0.257		6.305
			19	1	10.147	17	1	1.709	17	1	0.272		6.277
			20	1	10.228	18	1	1.726	18	1	0.274		6.281
			21	1	11.261	19	1	2.044	19	1	0.331		6.168
			22	1	11.347	20	1	2.321	20	1	0.375		6.177
			23	2	11.796	21	2	2.354	21	2	0.379		6.193
			25	1	12.000	23	1	2.501	23	1	0.411		6.084
			26	1	13.893	24	2	2.594	24	2	0.423		6.131

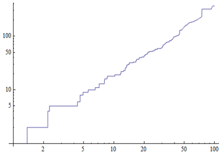
MC Eigenvalue Counting Function and Weyl Ratio

Eigenvalue Counting Function

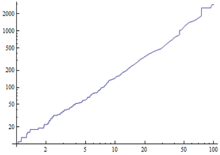
Level 2



Level 3

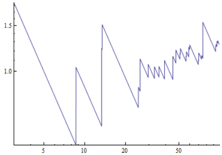


Level 4

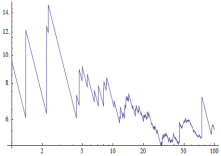


Weyl Ratios

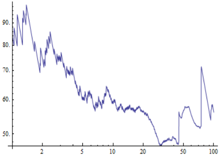
Level 2



Level 3

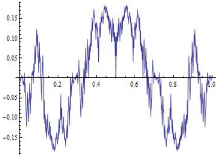


Level 4

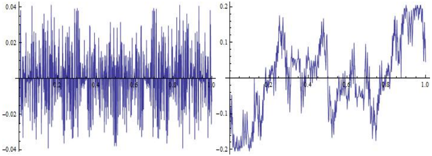


MC Eigenfunctions

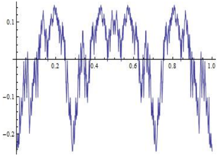
Eigenfunction # 2



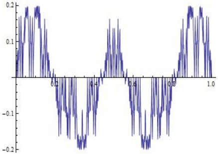
Eigenfunction # 3 & 4



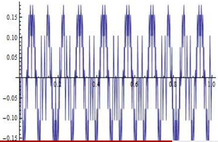
Eigenfunction # 5



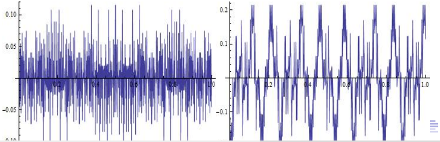
Eigenfunction # 6



Eigenfunction # 18

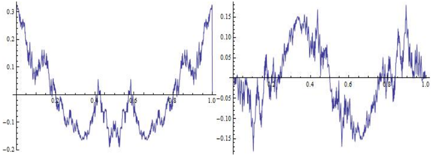


Eigenfunction #27 & 28

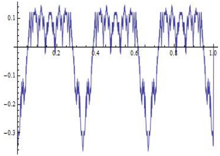


Triangle Eigenfunctions

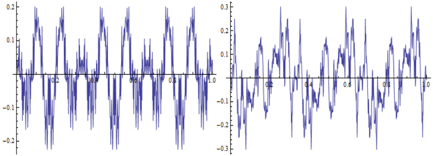
Eigenfunction # 2 & 3



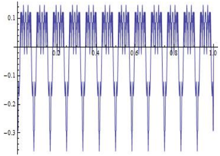
Eigenfunction # 4



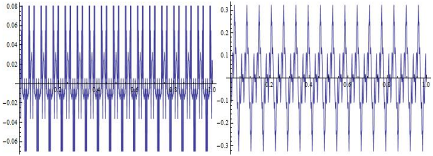
Eigenfunction # 9 & 10



Eigenfunction #11



Eigenfunction # 14 & 15



Acknowledgements

Thank you to the National Science Foundation funded Research Experience for Undergraduates.

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