Quantum Gravity, Asymptotic Safety, and Fractals

Martin Reuter

• A major success :

Classical General Relativity, based upon the Einstein - Hilbert action $S[g_{\mu\nu}] = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left(-R + 2\Lambda\right)$ is a highly successful effective field theory on length scales $\ell \gg \ell_{Pl} \equiv G^{1/2} \approx 10^{-33}$ cm .

• A natural question :

Is it possible to promote G.R. to a fundamental (microscopic) quantum theory of the gravitational interaction and spacetime structure, valid at arbitrarily small ℓ ?

• A first attempt :

The methods that work well for the electroweak + strong interactions fail :

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{(8\pi G)^{1/2} h_{\mu\nu}}_{\text{expand}}$$

Non - renormalizable in perturbation theory !

Increasing pert. order $\Rightarrow$
increasing # of divergences, i.e. counter terms, i.e. undetermined parameters

The Options:

- Leave the framework of Quantum Field Theory:

  LQG, Spin Foams, String Theory, ...

- Stay within (non-perturbative!) QFT:

  Asymptotic Safety

continuum approach:

Functional Renormalization Group Equation (FRGE),

Effective Average Action

statistical field theory:

Dynamical triangulations, Regge calculus, ...

The fundamental problem:

Give a meaning to ("define", "renormalize", "take the continuum limit of", ...) a functional integral over all metrics on a spacetime $\mathcal{M}$:

$$\int \mathcal{D}\hat{g}_{\mu\nu}\, e^{-S[\hat{g}_{\mu\nu}]}$$

S: diff $(\mathcal{M})$-invariant bare action,

e.g. S_{EH} + counter terms

$$\mathcal{D}\hat{g}_{\mu\nu} \equiv \prod_{x\in\mathcal{M}} \prod_{\mu,\nu} dg_{\mu\nu}(x)$$

requires regularization (UV Cutoff)

The strategy:

MR, 1996

Define and compute the functional integral indirectly by means of the associated Effective Average Action (EAA):

$\Gamma_k[g_{\mu\nu}, \cdots]$, one-parameter family of action functionals,
$0 \leq k < \infty$.

The problem, reformulated:

Construct <u>fully</u> <u>extended</u> integral curves
("RG trajectories") $k \longmapsto \Gamma_k[\cdot]$, $0 \leq k < \infty$
of an infinite dimensional flow $(\mathcal{T}, \beta)$.

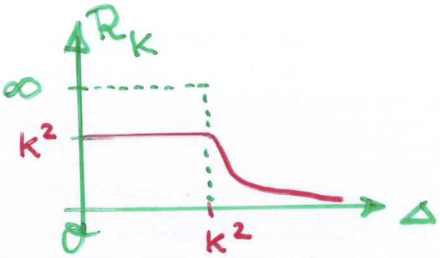
$\mathcal{T}$: "theory space" $\ni A[g_{\mu\nu}, \cdots]$
specified by field contents and symmetries

β : vector field on $\mathcal{T}$ defined by the functional renormalization group equation (FRGE) satisfied by the EAA:

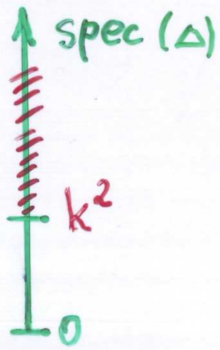
$$k\,\partial_k \Gamma_k = \beta(\Gamma_k)$$

The Effective Average Action

$$e^{W_k[J]} := \int \mathcal{D}\hat{\phi}\, e^{-S[\hat{\phi}]} \cdot e^{\int dx\, J\hat{\phi}} \cdot e^{-\frac{1}{2}\int \hat{\phi}\, R_k(\Delta)\, \hat{\phi}}$$



suppresses low eigenvalue ("low momentum", "large wave length") eigen-modes of Δ:
IR cutoff at (mass) scale $k \in [0, \infty)$



$$\Gamma_k[\phi] := \Big(\text{Legendre transf. of } W_k[J]\Big) - \frac{1}{2}\int \phi\, R_k(\Delta)\, \phi$$

Interpolating property:

$$\Gamma \xleftarrow{k \to 0} \Gamma_k \xrightarrow{k \to \infty} \sim S$$

ordinary eff. action   bare action

Functional Renormalization Group Equation (FRGE):

$$\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_k R_k\right]$$

Effective field theory properties of Γ_k :

- $\Gamma_k[\phi]$ is generating fctl. for correlators of fields which are averaged over space time volumes of size $l \approx k^{-1}$.

- The effective field equation

$$\frac{\delta \Gamma_k}{\delta \phi(x)}\left[\langle\phi\rangle_k\right] = 0$$

yields expectation value (mean field)

$\langle\phi\rangle_k$ ≡ field "seen" in experiment with "microscope" having resolving power l

- For observable $\mathcal{O}(\hat{\phi})$ involving only momenta near k :

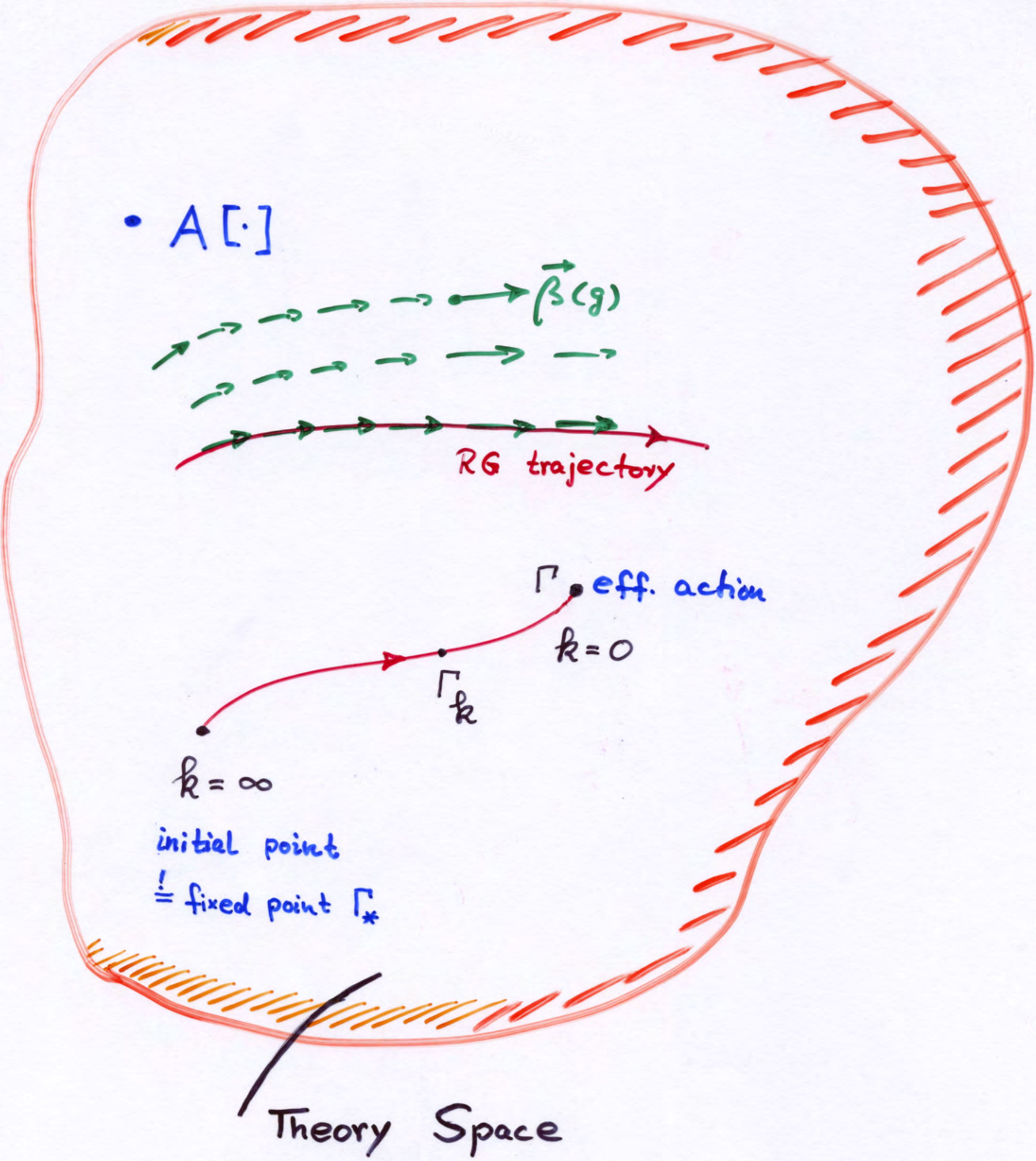
$$\langle \mathcal{O}(\hat{\phi}) \rangle \approx \mathcal{O}\left(\langle\phi\rangle_k\right)$$

The Asymptotic Safety idea:

- Take the infinite-cutoff limit of an UV-regularized quantum theory of gravity at a non-trivial RG fixed point with a finite dimensional UV-critical hypersurface, assuming it exists.

- The resulting continuum theory is predictive and well behaved at arbitrarily short distances.

(S. Weinberg, 1979, 2009)



$A[\cdot]$
$\vec{\beta}(g)$
RG trajectory
Γ eff. action
$k=0$
Γ_k
$k=\infty$
initial point
$\overset{!}{=}$ fixed point Γ_*
Theory Space

The Einstein-Hilbert Truncation

MR, 1996

Ansatz:

$$\Gamma_k = -\frac{1}{16\pi G_k}\int d^dx \sqrt{g}\,\left(R - 2\Lambda_k\right)$$

+classical gauge fixing and ghost terms

Running coupling constants:

Newton constant G_k , dimensionless: $g(k) = k^{d-2} G_k$

cosmological constant Λ_k , dimensionless: $\lambda(k) = k^{-2}\Lambda_k$

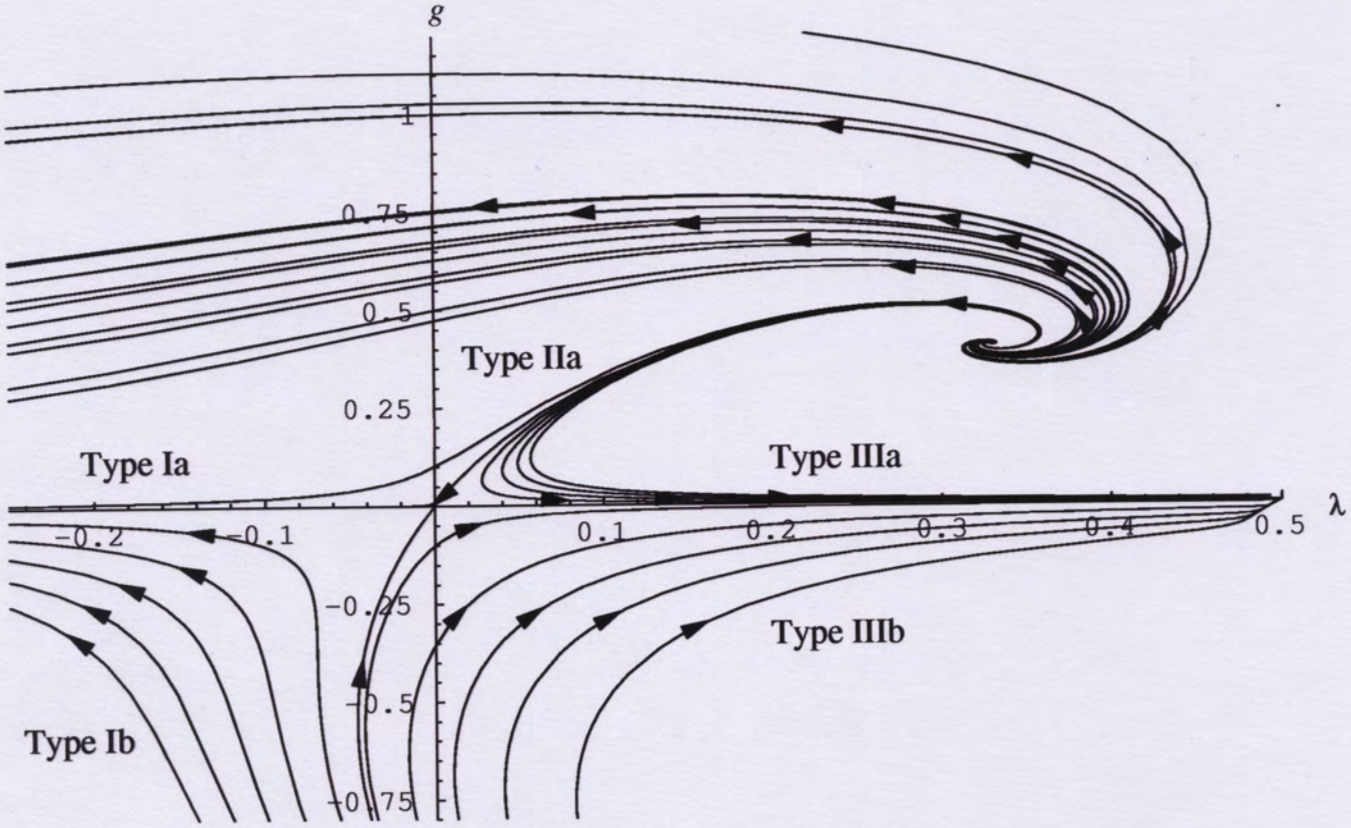
Insert ansatz into FRGE, "project out" monomials retained:

$$\mathrm{Tr}[\cdots] = (\cdots)\int\sqrt{g} + (\cdots)\int\sqrt{g}R + \cdots \quad \Rightarrow$$

$$\boxed{\begin{aligned} k\,\partial_k\, g(k) &= \beta_g(g,\lambda) \\ k\,\partial_k\, \lambda(k) &= \beta_\lambda(g,\lambda) \end{aligned}}$$

Einstein - Hilbert Truncation :

RG Flow on the $g-\lambda$ plane



M.R., F. Saueressig, hep-th/0110054

Anomalous dimension η_N :

$$k \partial_k g = \beta_g \equiv \left[(d-2) + \underbrace{\eta_N (g, \dots)}_{\sim \hbar} \right] \cdot g$$

with $$\boxed{\eta_N \equiv k \partial_k \log G_k}$$

- Interpretation: $$\boxed{d_{eff} = d + \eta_N}$$

$$\Gamma_k \sim \frac{1}{G_k} \int d^d x \sqrt{g} \; R \quad \Longrightarrow$$

Graviton propagator (on flat space)

$$\langle h_{\mu\nu}(x) \, h_{\rho\sigma}(y) \rangle \sim \frac{1}{|x-y|^{d_{eff}-2}} \qquad \text{if} \quad d_{eff} \neq 2$$

$$\sim \log |x-y| \qquad \text{if} \quad d_{eff} = 2$$

- At a non-Gaussian fixed point (NGFP):

$$\beta_g = 0 \quad \Longleftarrow \quad \eta_N = (2-d) \equiv \eta_*$$

Dynamical dimensional reduction:

$$d \longrightarrow d^*_{eff} \equiv d + \eta_* = 2$$

Metrics on QEG Spacetimes

- Fix a quantum theory by picking a specific RG trajectory $k \mapsto \Gamma_k[\cdot]$
- Solve eff. field eq. at any k:

$$\frac{\delta \Gamma_k}{\delta g_{\mu\nu}}\left[\langle g \rangle_k\right] = 0$$

scale dependent mean field

- A single trajectory gives rise to infinitely many "on-shell metrics":

$$\left\{ \langle g_{\mu\nu}(x) \rangle_k \;\middle|\; k = 0, \cdots, \infty \right\}$$

- Metric structure of "QEG spacetime" is described by <u>infinitely many</u> classical Riemannian metrics.

<u>Interpretation</u>: Observing spacetime under a microscope of resolving power ℓ_1 one sees a classical manifold with metric

$$\langle g_{\mu\nu} \rangle_{k_1} \quad \text{where} \quad \ell_1 = \ell(k_1)$$

- Metric is scale dependent!

<u>Analogy</u>: The length of the coast line of England depends on the size of the yardstick used to measure it.

Regimes with power law running

$$\langle g_{\mu\nu} \rangle_k = \left(\frac{k_0}{k}\right)^{\delta} \langle g_{\mu\nu} \rangle_{k_0}$$

- *classical regime:* $G_k, \Lambda_k, \ldots = \text{const}$, $\boxed{\delta = 0}$

- *Non-Gaussian fixed point regime:* $\boxed{\delta = 2}$

$$G_k = g_* k^{2-d}, \; \Lambda_k = \lambda_* k^2, \ldots$$

$\Longrightarrow$ *self similarity:*

Proper length of any object equals the resolving power of the "microscope" used to measure it,

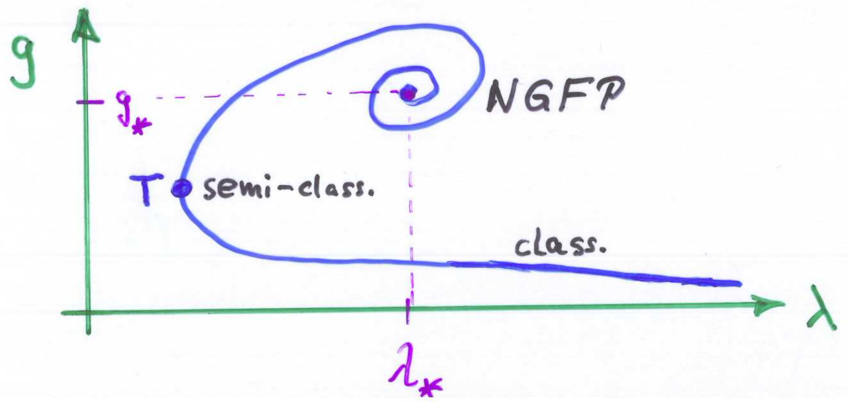
$$\left(\langle g_{\mu\nu} \rangle_k \, \Delta x^\mu \Delta x^\nu \right)^{1/2} \sim k^{-1} \sim \ell$$

- *Semi-classical regime:* $\boxed{\delta = d}$

not universal, eg. if: no matter fields, EH truncation

$$R_{\mu\nu}(g) = \frac{2}{2-d} \underbrace{\Lambda_k}_{\sim k^d} g_{\mu\nu} \quad \Longrightarrow \quad \langle g_{\mu\nu} \rangle_k \sim k^{-d}$$

Depends on trajectory in general!



Diffusion of test particles on QEG spacetimes

O. Lauscher, MR, 2005
MR, F. Saueressig, 2011

quantum-corrected diffusion equation:

$$\left[\frac{\partial}{\partial T} - \langle \Delta_g \rangle \right] K(x, x'; T) = 0$$

- Evaluate expectation value by exploiting the effective field theory properties of Γ_k.
- Comparing to

$$K(x, x'; T) = \frac{1}{t^{d_s/2}} \, \Phi\left(\frac{\mathrm{dist}_{g_0}(x, x')}{t^{1/d_w}} \right)$$

yields, when $\langle g_{\mu\nu} \rangle_k \sim k^\delta$:

Spectral dimension: $\mathcal{D}_s = \dfrac{2d}{2+\delta}$

Walk dimension: $\mathcal{D}_w = 2 + \delta$

Hausdorff dimension: $\mathcal{D}_H = d$

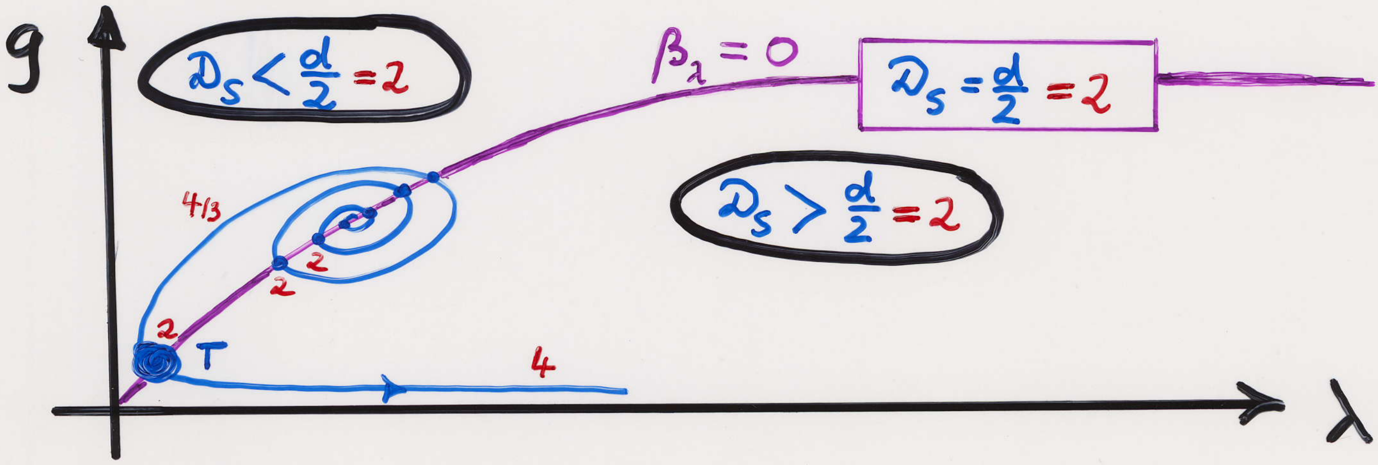
(Satisfy $\mathcal{D}_s \mathcal{D}_w = 2 \mathcal{D}_H$)

Classical		Semi-classical		NGFP	
$\delta = 0$		$\delta = d \quad (d \neq 3)$		$\delta = 2$	
$\mathcal{D}_S = d$	$= 4$	$\mathcal{D}_S = \frac{2d}{2+d}$	$= \frac{4}{3}$	$\mathcal{D}_S = \frac{d}{2}$	$= 2$
$\mathcal{D}_W = 2$	$= 2$	$\mathcal{D}_W = 2 + d$	$= 6$	$\mathcal{D}_W = 4$	$= 4$
$\mathcal{D}_H = d$	$= 4$	$\mathcal{D}_H = d$	$= 4$	$\mathcal{D}_H = d$	$= 4$
$d_{eff} = d$	$= 4$	$d_{eff} \approx d$	$= 4$	$d_{eff} = 2$	$= 2$

Scale dependent spectral dimension

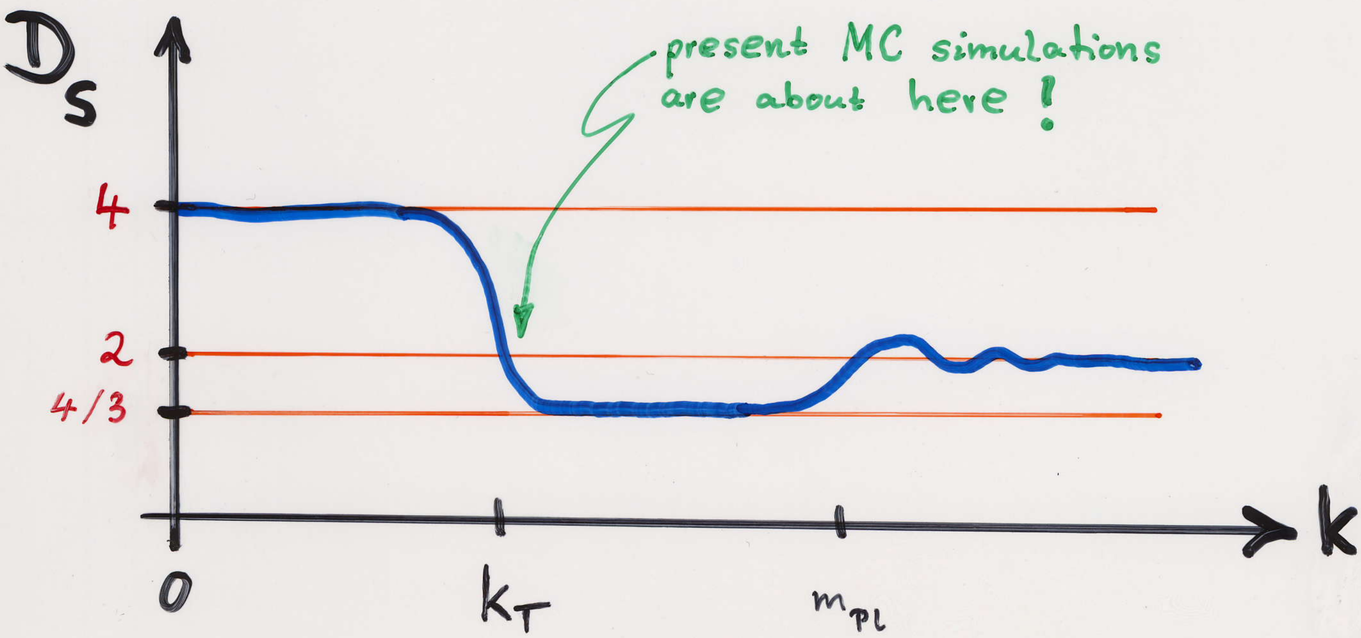
$$\Lambda_k \sim k^{\delta(k)} \quad \Rightarrow \quad \mathcal{D}_s = \frac{2d}{2+\delta}$$

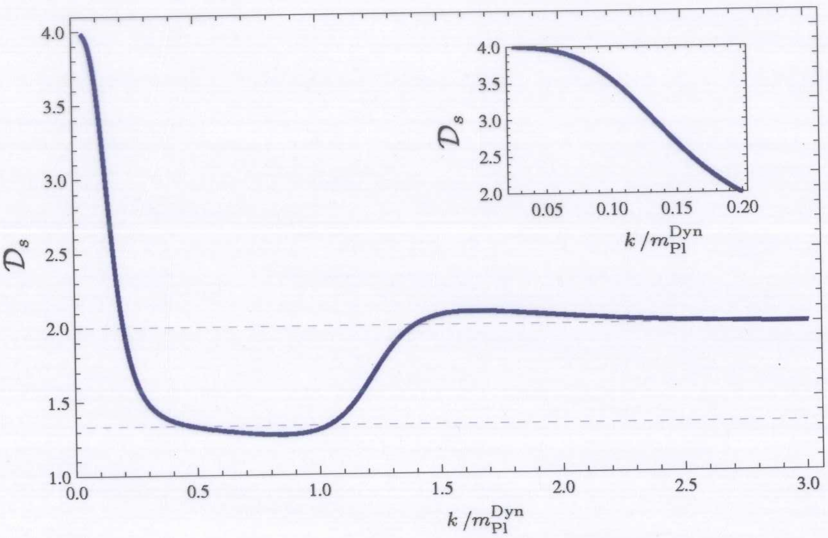
classical	semiclass.	NGFP
$\delta = 0$	$\delta = d \quad = 4$	$\delta = 2$
$\mathcal{D}_s = d = 4$	$\mathcal{D}_s = \dfrac{2d}{2+d} = \dfrac{4}{3}$	$\mathcal{D}_s = \dfrac{d}{2} = 2$

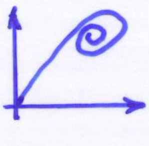


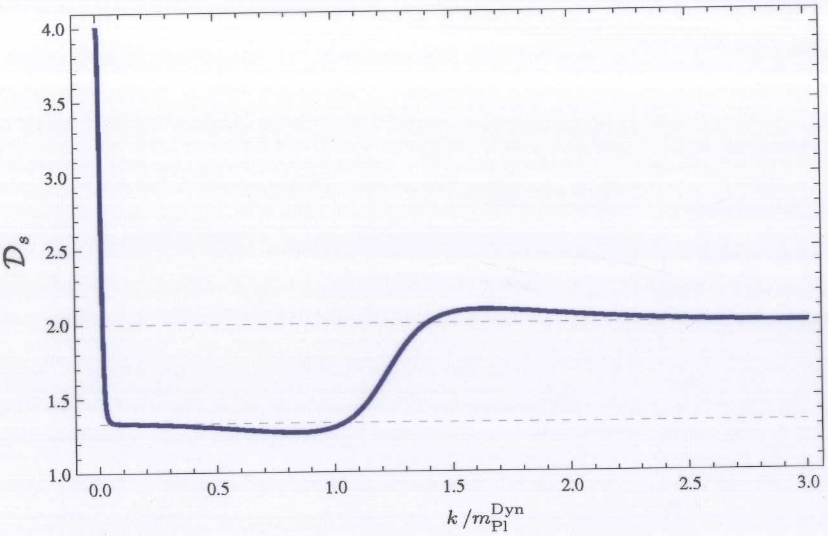
$\rightsquigarrow$ infinitely many points with $\mathcal{D}_s = \frac{d}{2}$!

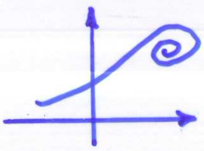
When T $\rightarrow$ origin, 3 plateaus arise :

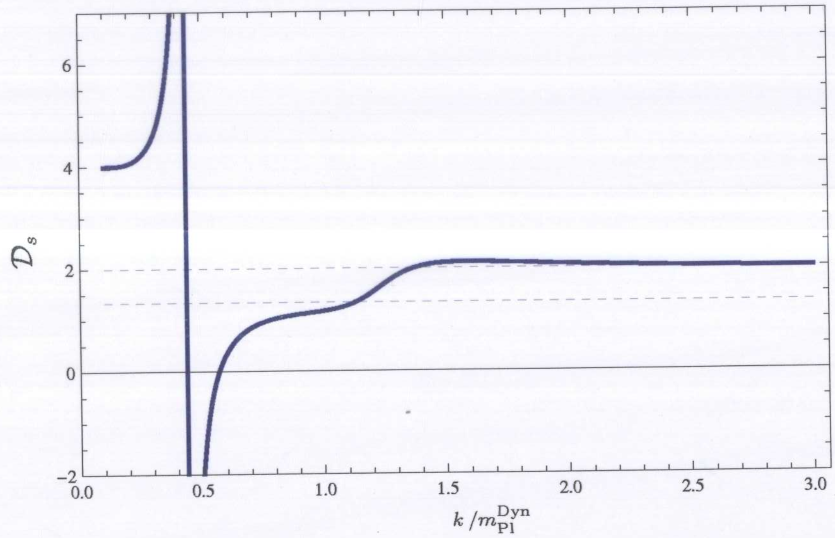












D. Becker, MR, arXiv: 1404.4537

- **CDT simulations:**

$$D_s = (4.02 \pm .01) \longrightarrow (1.80 \pm 0.25)$$

Ambjørn, Jurkiewicz, Loll; 2005

$$D_s = (3.05 \pm .04) \longrightarrow (2.04 \pm 0.10)$$

Benedetti, Henson; 2009

- **EDT simulation:**

$$D_s = (4.04 \pm 0.26) \longrightarrow (1.457 \pm 0.064)$$

Laiho, Coumbe, 2011

- **LQG (spatial section) and Spin-Foams:**

$$D_s = 3 \longrightarrow 1.5 \longrightarrow (1.5 \text{ or } 2)$$

$$D_s = 4 \longrightarrow 2$$

Modesto, 2008

3D Spin-Foams (P.R. / T.V.):

$$D_s = 3 \longrightarrow 1.5 \longrightarrow 2$$

Modesto, Caravelli, 2009

- **Strong Coupling limit of W-dW. eq.:** $4 \rightarrow 2$

Carlip, 2009

- **Models:** **Quantum sphere, κ-Minkowski:**

$$D_s = 4 \longrightarrow (< 4)$$

Benedetti, 2008

QFT on given fractal

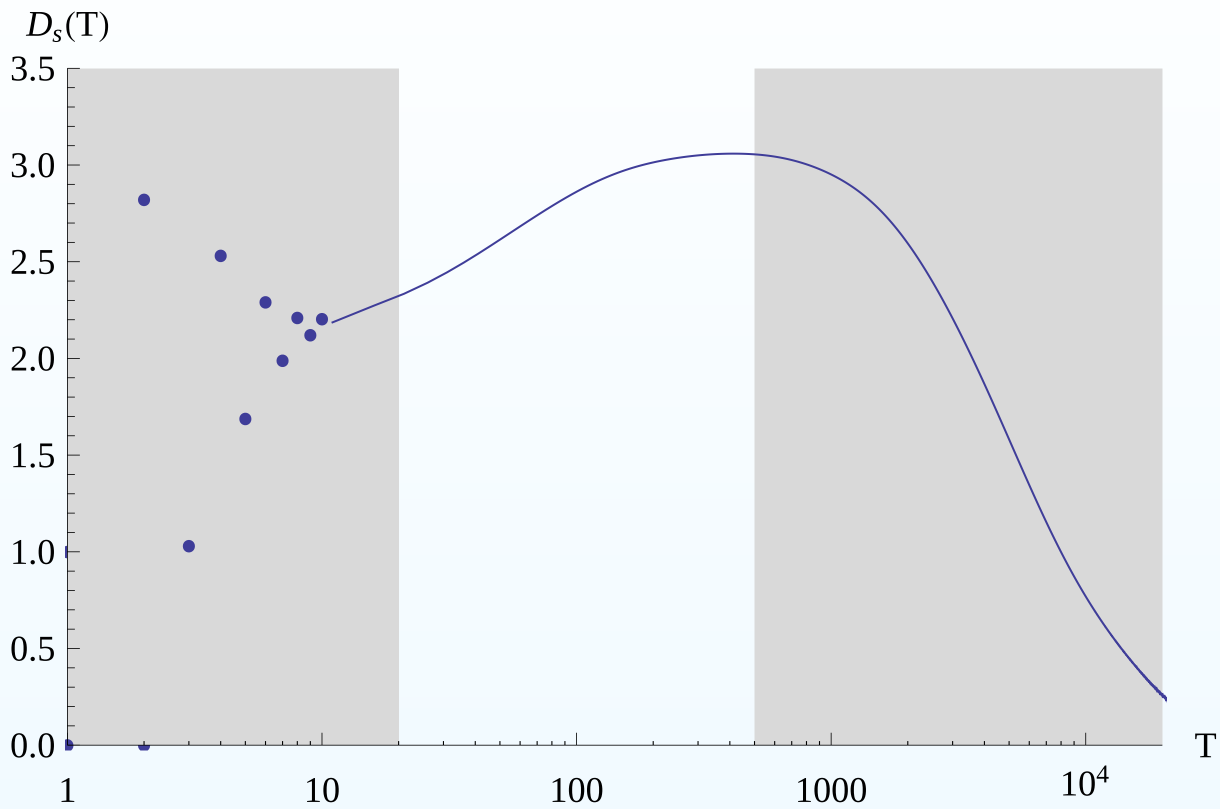
Calcagni, 2009

How is it possible that numerical simulations,
still far from the continuum limit,
do already observe a dimensional reduction ?

MR, F. Saueressig (2011)

Spectral Dimension measured in 3-dimensional CDT

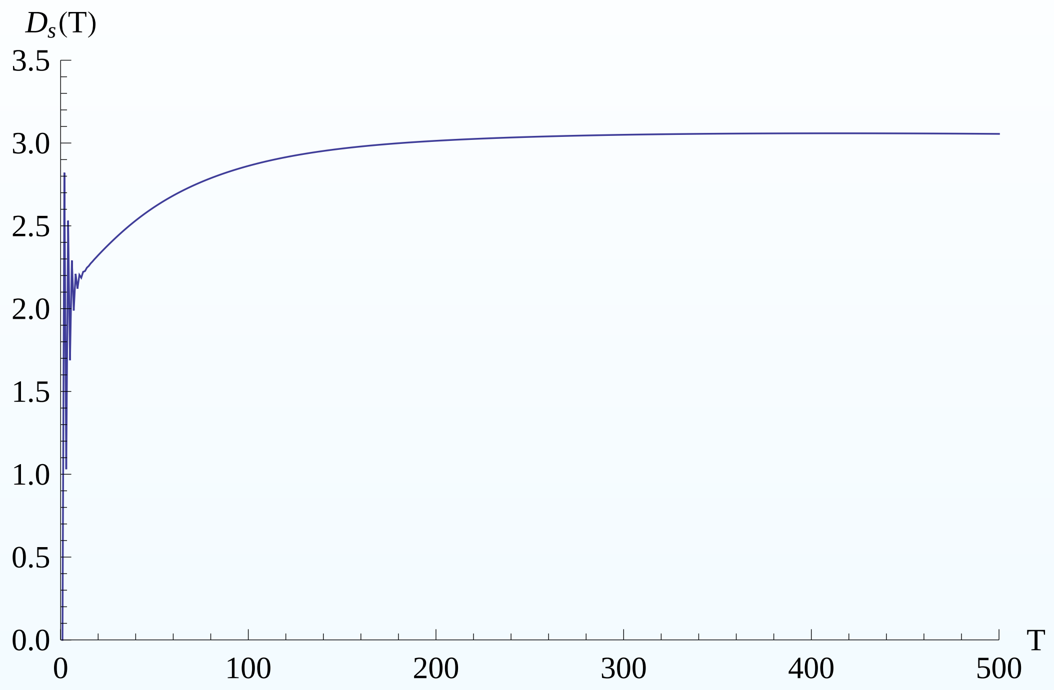
[D. Benedetti, J. Henson, Phys. Rev. D 80 (2009) 124036]



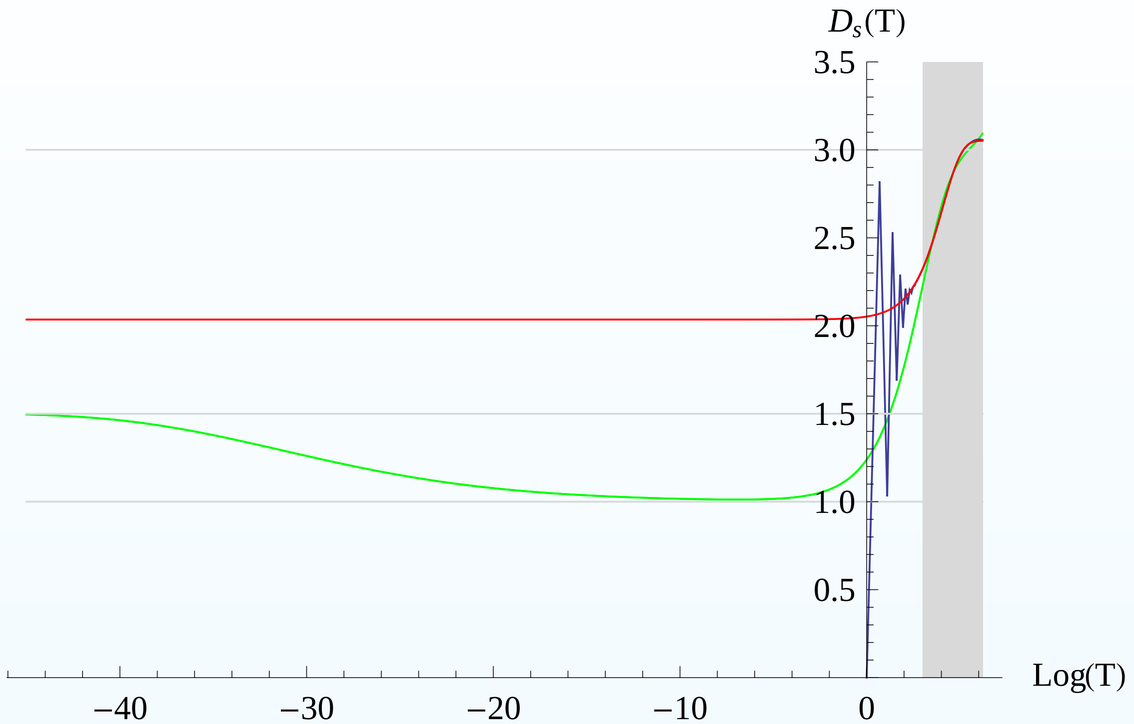
	T	≤ 20	oscillations (discrete simplex structure)
$20 \leq$	T	≤ 500	good data
$500 \leq$	T		exponential fall-off (triangulation is compact)

Determining the spectral dimension in CDT

[D. Benedetti, J. Henson, Phys. Rev. D 80 (2009) 124036]



Comparing spectral dimensions in $d = 3$



- CDT and QEG agree with data within 1% accuracy
- no data-points on the semi-classical and NGFP-plateau

resolves puzzle between CDT data and QEG prediction!

Conclusion

- The scale dependent spectral dimension seems to be an ideal diagnostic tool in order to monitor the approach to the continuum limit in numerical simulations.

- Its observation in real Nature seems extremely difficult though:

  - $\mathcal{D}_s$ refers to diffusion of <u>test</u> particles!
    idealization: should include backreaction

  - Pure and matter coupled gravity are different;
    fractal features require

    $$\Lambda_K \, g_{\mu\nu} \gg 8\pi G \, T_{\mu\nu}^{\text{Matter}}$$

    or strongly K-dependent $T_{\mu\nu}^{\text{Matter}}$

  - Related to (highly nonperturbative) problem of vacuum structure (cf. QCD)

  - Needed: fully dressed inv. graviton / photon propagator