Differential Equations on Cubic Julia Sets

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Given a complex polynomial $f$, the associated filled-in Julia set is the set of points $z$ such that the sequence $z, f(z), f(f(z)), \ldots, f^n(z), \ldots$ does not diverge to infinity.

The Julia set is the boundary of the filled-in Julia set.

A Julia set is called quadratic or cubic if its corresponding polynomial is quadratic or cubic.
Julia Sets

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The standard approach to defining Laplacians

- Define a sequence of finite graphs $X_m$ approximating the fractal.
- Define a graph energy on each graph by
  \[ E_m(u) = \sum_{\text{edges } (x,y) \text{ in } X_m} c_{x,y} (u(x) - u(y))^2 \]
- Hope for a choice of conductances such that $E_{m+1}(\tilde{u}) = E_m(u)$ for all $m, u$.
- Define the energy $E = \lim_{m \to \infty} E_m$
- Use the energy to obtain a Laplacian $\Delta u$ by the weak formulation: $E(u, v) = -\int (\Delta u) v$ for any $v$. 

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For quadratic (cubic) Julia sets, there is a map from the circle to the Julia set intertwining doubling (tripling) on the circle with the action of $P$ on the Julia set.
We use this to represent the Julia Set as a circle modulo identifications.
Julia Set as Circle with Identifications

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For each Julia set, there is a choice of a partition of the circle into three equally-sized regions A, B, C.

Each point on the circle has a kneading sequence with respect to A, B, and C, recording the point's orbit.

We identify points that have the same kneading sequence* and some other conditions.
The Graph Approximation

- For any finite set of points on the circle (mod identifications), form a graph by adding an edge between neighboring points.
- Start with an initial set \( X_0 \)
- Let \( X_{m+1} := \{ x : 3x \in X_m \} \)
Subdivision Rules

For each Julia set, we express the fractal structure of the Julia set by finding subdivision rules:

- We classify the intervals into finitely many “types”
- For each type, we describe how it subdivides in the next level in terms of the other types
Let \( n \) be the number of types involved in the subdivision rules.

Define \( E_m^{(k)}(u) = \sum_{i: (t_i, t_{i+1}) \text{ is of type } k} \frac{(u(t_{i+1}) - u(t_i))^2}{|u(t_{i+1}) - u(t_i)|} \).

Define \( E_m(u) = \sum_{k=1}^{n} b_k E_m^{(k)}(u) \) for some choice of constants \( b_1, \ldots, b_n \), for all \( m \).

Look for \( b_k \) such that there exists an \( r \) with \( E_{m+1}(\tilde{u}) = r E_m(u) \) for all \( m, u \).

Define \( \mathcal{E} = \lim_{m \to \infty} r^{-m} E_m \).

\( \mathcal{E} \) is self-similar: \( \mathcal{E}(u \circ P) = \frac{9}{r} \mathcal{E}(u) \)
The problem of solving for the $b_k$ and $r$ can be viewed either as an eigenvector-eigenvalue problem or as a system of resistance problems.

If all subdivisions are internal or in pairs, solving for the $b_k$ and $r$ is simple.
Example 1

- \( A = \left( \frac{1}{24}, \frac{9}{24} \right], B = \left( \frac{13}{24}, \frac{21}{24} \right], C = \text{the rest.} \)
- \( P(z) = z^3 + \frac{3}{\sqrt{2}}z \)
Example 1

Using $X_0 = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$, the subdivision rules are:

\[
\begin{align*}
0 & \rightarrow 1 \\
1 & \rightarrow 0 1 \\
2 & \rightarrow 3 0 \\
3 & \rightarrow 3 3 \\
\end{align*}
\]
Example 1

Weak pairs are bad. No solution to resistance problem unless we add an extra edge.
Example 1

Solution: Procrastination. "Why do today what you could leave till tomorrow?"

All ways of procrastinating yield the same energy, and the energy is self-similar.
Example 2

- $A = (\frac{6}{24}, \frac{14}{24}], B = (\frac{17}{24}, \frac{1}{24}], C = \text{the rest.}$

- Using $X_0 = \{\frac{2}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}\}$, we are unable to classify subdivision rules using finitely many types.
Example 2

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Solution: the Julia set is begging you to add more points to $X_0$: the limit identifications.
We hope that, in general, any “bad” subdivision data can be turned into “good” subdivision data by using either of these two fixes: procrastination and adding limit identifications to $X_0$. 
Even when the above methods work, sometimes the only solutions to the conductances $b_k$ are negative.

And sometimes the system of resistance problems is inconsistent, with no solutions.
Laplacian

- Use the weak formulation, using the standard Lebesgue measure on the circle.
- Using finite element method/ finite difference method, we can compute eigenfunctions and eigenvalues of the Laplacian.
Eigenfunctions for Example 1
Eigenfunctions for Example 1

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Eigenfunctions for Example 1

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Some References

- Aougab, Dong, Strichartz, *Laplacians on a family of Julia sets II*, Communications on pure and applied analysis, 12 (2013), 1-58