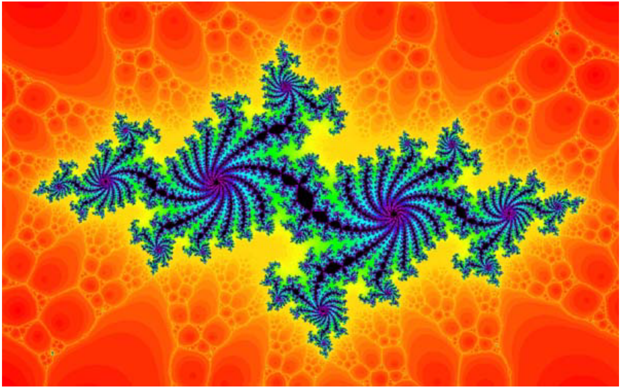
Differential Equations on Cubic Julia Sets

Yehonatan Sella

University of California Los Angeles

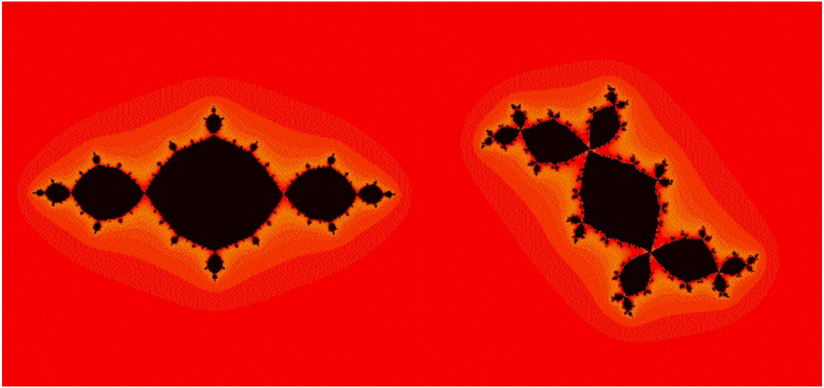
June 16, 2014

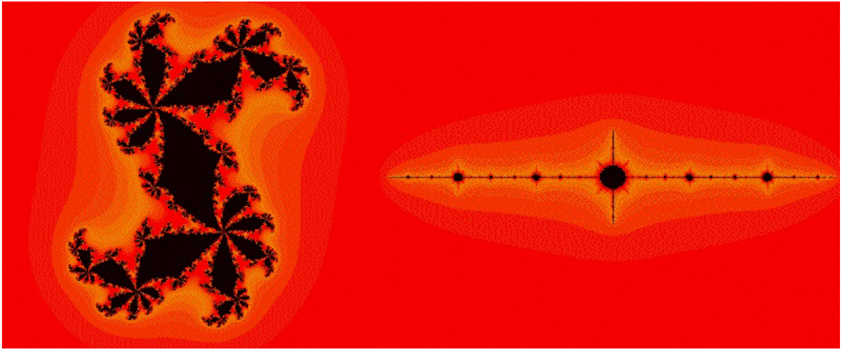


Julia Sets

- Given a complex polynomial f, the associated filled-in Julia set is the set of points z such that the sequence $z, f(z), f(f(z)), ..., f^{\circ n}(z), ...$ does not diverge to infinity.
- The Julia set is the boundary of the filled-in Julia set.
- A Julia set is called quadratic or cubic if its corresponding polynomial is quadratic or cubic.

Julia Sets





The standard approach to defining Laplacians

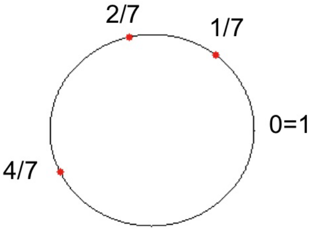
- Define a sequence of finite graphs X_m approximating the fractal.
- Define a graph energy on each graph by

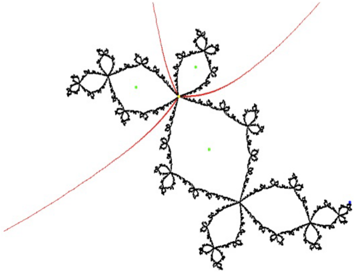
$$\mathcal{E}_m(u) = \sum_{\text{edges } (x,y) \text{ in } X_m} c_{x,y}(u(x) - u(y))^2$$

- Hope for a choice of conductances such that $\mathcal{E}_{m+1}(\tilde{u}) = \mathcal{E}_m(u)$ for all m, u.
- Define the energy $\mathcal{E} = lim_{m\to\infty}\mathcal{E}_m$
- Use the energy to obtain a Laplacian Δu by the weak formulation: $E(u, v) = -\int(\Delta u)v$ for any v.

External Ray Parameterization

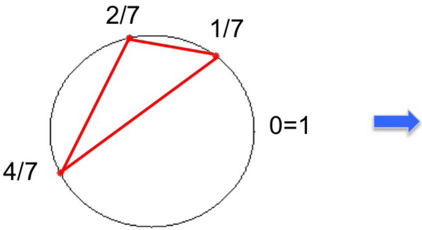
For quadratic (cubic) Julia sets, there is a map from the circle to the Julia set intertwining doubling (tripling) on the circle with the action of P on the Julia set.

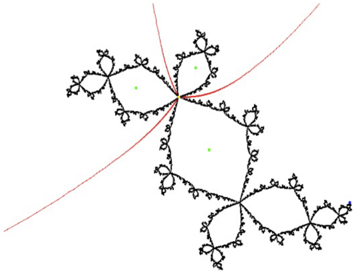




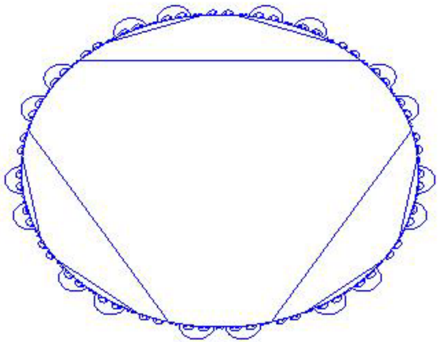
Julia Set as Circle with Identifications

We use this to represent the Julia Set as a circle modulo identifications.

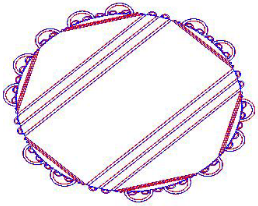


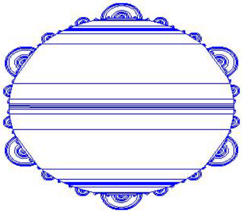


Julia Set as Circle with Identifications

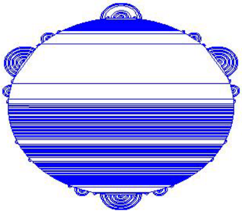


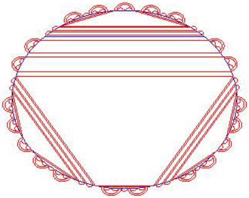
Julia Set as Circle with Identifications



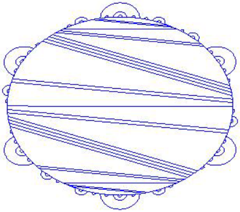


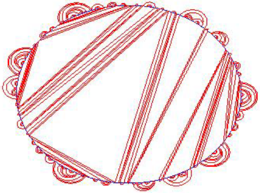
Julia Set as Circle with Identifications



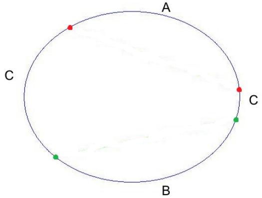


Julia Set as Circle with Identifications



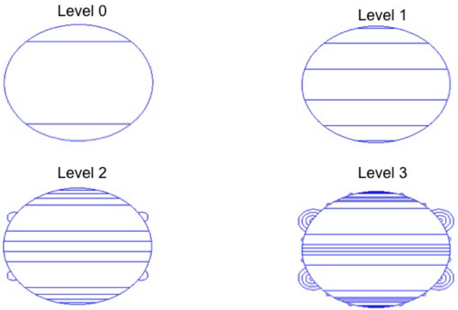


The Algorithm for Identifications



- For each Julia set, there is a choice of a partition of the circle into three equally-sized regions A,B,C
- Each point on the circle has a kneading sequence with respect to A,B and C, recording the point's orbit.
- We identify points that have the same kneading sequence*
  *and some other conditions.

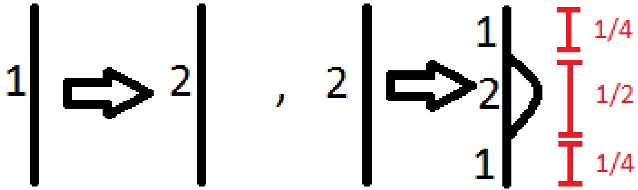
- For any finite set of points on the circle (mod identifications), form a graph by adding an edge between neighboring points.
- Start with an initial set X_0
- Let $X_{m+1} := \{x : 3x \in X_m\}$



Subdivision Rules

For each Julia set, we express the fractal structure of the Julia set by finding subdivision rules:

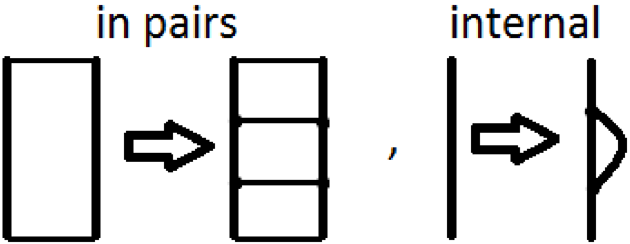
- We classify the intervals into finitely many "types"
- For each type, we describe how it subdivides in the next level in terms of the other types



Energy on Julia Sets

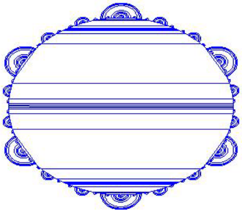
- Let n be the number of types involved in the subdivision rules.
  .
- Define $E_m^{(k)}(u) = \sum_{i:(t_i,t_{i+1}) \text{ is of type } k} \frac{(u(t_{i+1})-u(t_i))^2}{|u(t_{i+1})-u(t_i)|}$.
- Define $E_m(u) = \sum_{k=1}^{n} b_k E_m^{(k)}(u)$ for some choice of constants $b_1, \ldots, b_n$, for all m.
- Look for b_k such that there exists an r with $E_{m+1}(\tilde{u}) = rE_m(u)$ for all m, u.
- Define $\mathcal{E} = lim_{m\to\infty} r^{-m} E_m$.
- $\mathcal{E}$ is self-similar: $\mathcal{E}(u \circ P) = \frac{9}{r}\mathcal{E}(u)$

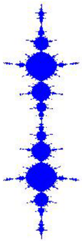
- The problem of solving for the b_k and r can be viewed either as an eigenvector-eigenvalue problem or as a system of resistance problems.
- If all subdivisions are internal or in pairs, solving for the b_k and r is simple.



Example 1

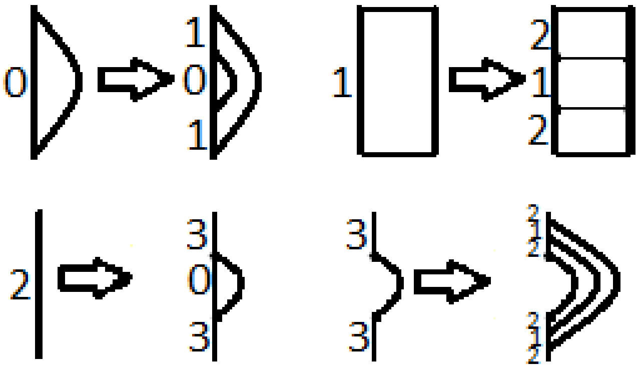
- $A = (\frac{1}{24}, \frac{9}{24}], B = (\frac{13}{24}, \frac{21}{24}], C =$ the rest.
- $P(z) = z^3 + \frac{3}{\sqrt{2}} z$





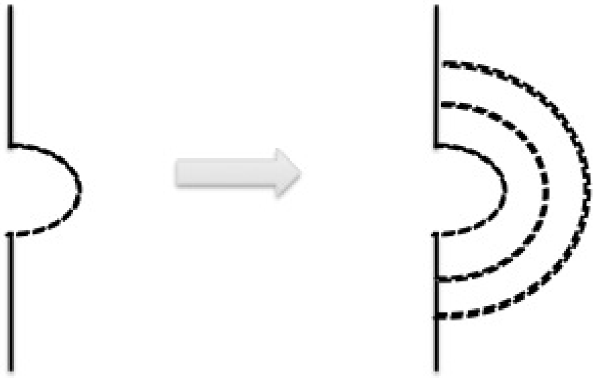
Example 1

Using $X_0 = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$, the subdivision rules are:



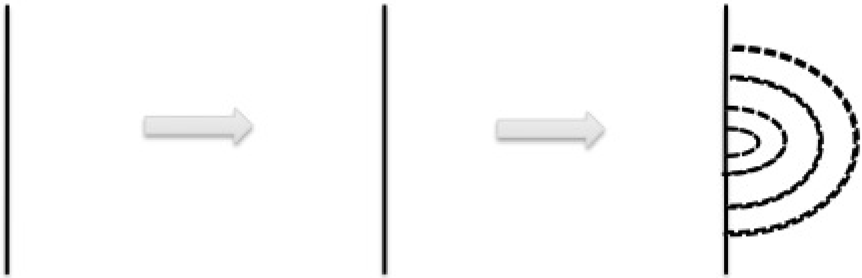
Example 1

Weak pairs are bad. No solution to resistance problem unless we add an extra edge.



Example 1

Solution: Procrastination. "Why do today what you could leave till tomorrow?"

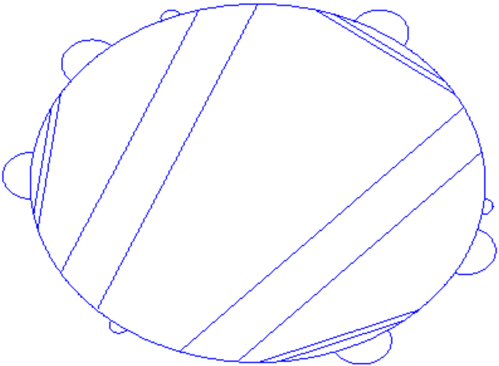


All ways of procrastinating yield the same energy, and the energy is self-similar.

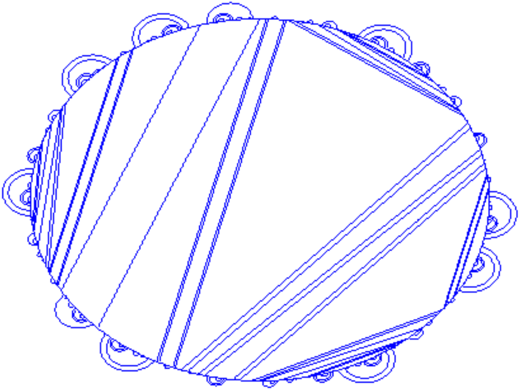
Example 2

- $A = (\frac{6}{24}, \frac{14}{24}]$, $B = (\frac{17}{24}, \frac{1}{24}]$, $C =$ the rest.
- Using $X_0 = \{\frac{2}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}\}$, we are unable to classify subdivision rules using finitely many types

Example 2

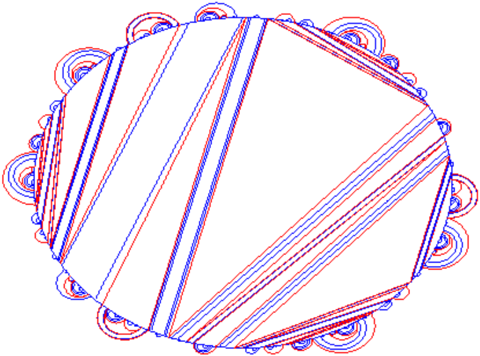


Example 2



Example 2

Solution: the Julia set is begging you to add more points to X_0: the limit identifications.



We hope that, in general, any "bad" subdivision data can be turned into "good" subdivision data by using either of these two fixes: procrastination and adding limit identifications to X_0.

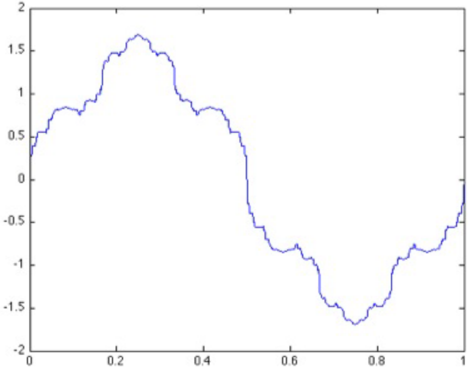
More Serious Problems

- Even when the above methods work, sometimes the only solutions to the conductances b_k are negative.
- And sometimes the system of resistance problems is inconsistent, with no solutions

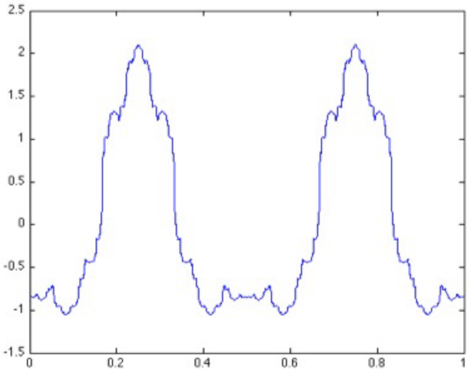
Laplacian

- Use the weak formulation, using the standard Lebesgue measure on the circle
- Using finite element method/ finite difference method, we can compute eigenfunctions and eigenvalues of the Laplacian

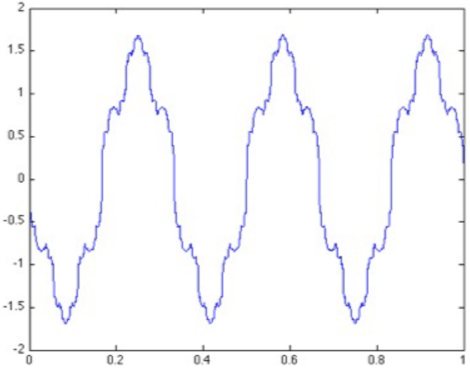
Eigenfunctions for Example 1



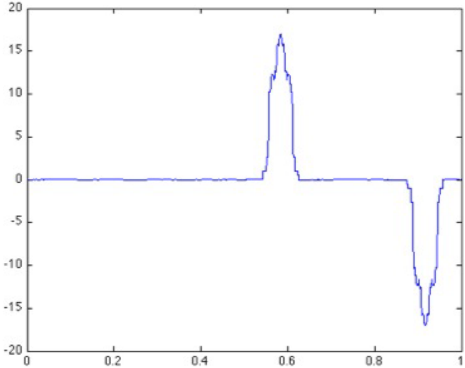
Eigenfunctions for Example 1



Eigenfunctions for Example 1



Eigenfunctions for Example 1



Some References

- Aougab, Dong, Strichartz, *Laplacians on a family of Julia sets II*, Communications on pure and applied analysis, 12 (2013), 1-58
- Spicer, Strichartz, Totari, *Laplacians on Julia sets III: cubic Julia sets and formal matings*, Contemporary Mathematics 600 (2013), 327-348
- Poirier, *On Post Critically Finite Polynomials Part One: Critical Portraits*, Fund. Math. 203 (2009) #2, 107-163