Fractal Geometry and Complex Dimensions in Metric Measure Spaces

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Overview

Main References

FGCD M.L. Lapidus, M. Frankenhuijsen: *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings, Second Edition (of the 2006 First Edition)*, Springer Monograph in Mathematics, Springer, New York, 2013, 593 pages.

FZF M.L. Lapidus, G. Radunović, D. Žubrinić: *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*, research monograph, preprint, 317 pages. Expected submission date: June 2014.

Metric Measure Spaces

Metric Measure Space

A **Metric Measure Space** (or MM space) is a set X equipped with a metric d and a positive Borel measure μ that is doubling; $\exists C$ positive such that

$$\mu(B_d(x, 2r)) \leq C\mu(B_d(x, r)).$$

Metric Measure Spaces

Given a metric, there is always a measure that we can construct on the space:

Hausdorff Measure

For any s nonnegative, $A \subset X$, define the s-**dimensional Hausdorff outer measure**:

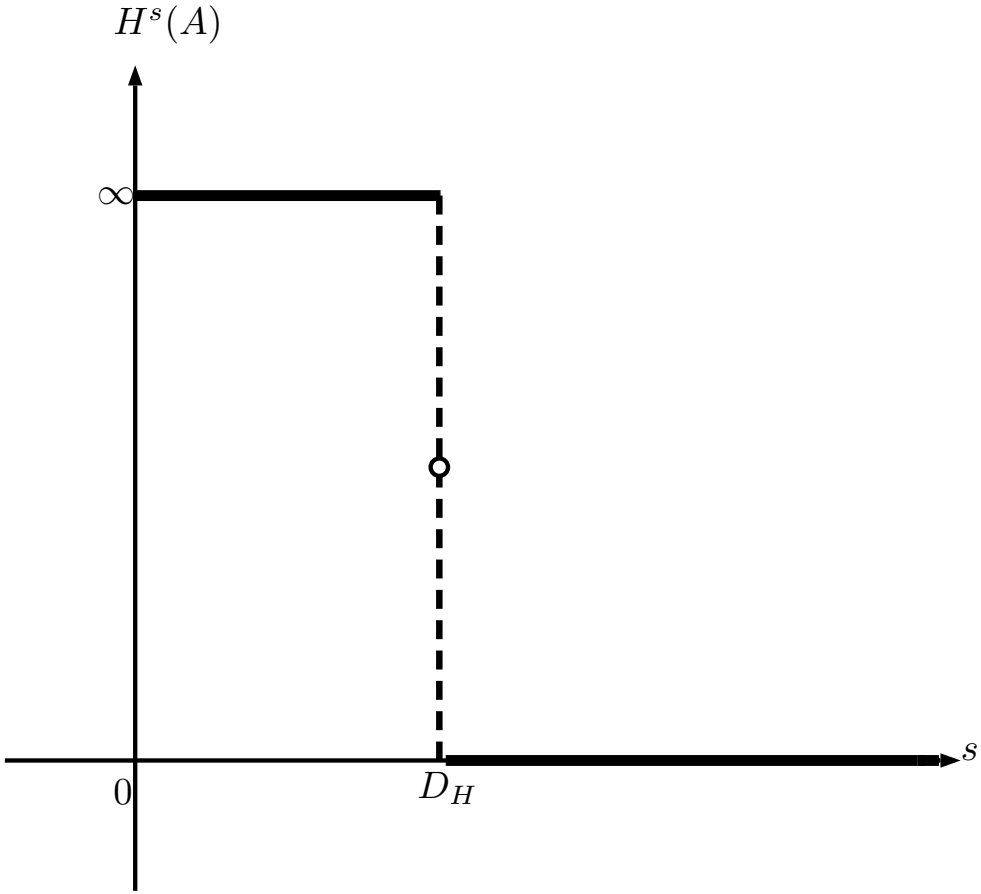
$$H^s(A) = \lim_{\delta \to 0} H^s_\delta = \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} (\mathrm{diam} E_j)^s : A \subset \bigcup_{j=1}^{\infty} E_j, \mathrm{diam} E_j < \delta \right\}.$$

A set A has **Hausdorff dimension** D_H if

$$\begin{aligned} D_H &= \inf\{s \geq 0 : H^s(A) = 0\} \\ &= \sup\{s \geq 0 : H^s(A) = \infty\}. \end{aligned}$$

We call H^D the **D-dimensional Hausdorff measure** when restricted to Borel sets.

Metric Measure Spaces



An example of the s-dimensional Hausdorff measure of a set A with Hausdorff dimension D_H .

Metric Measure Spaces

MM spaces are a growing area of research, especially in fields such as:

- Harmonic Analysis
- Partial Differential Equations
- Probability Theory
- Function Spaces
- Geometric Analysis on Non-Smooth Spaces
- Analysis on Fractals

Metric Measure Spaces

We will see that we need a stronger requirement than just the doubling condition:

Ahlfors regularity

A MM space is **Ahlfors regular of dimension D** (here on, **regular**) if $\exists K > 0$ such that

$$K^{-1} r^D \leq \mu(B(x,r)) \leq K r^D$$

$\forall x \in X, 0 < r \leq \text{diam} X$.

If only the upper (resp. lower) bounds are satisfied, we call the space **upper (resp. lower) Ahlfors regular of dimension D**.

Metric Measure Spaces

The measure μ of a regular D dimensional space and H^D are equivalent in that there is a constant C depending only on K such that

$$C^{-1}\mu(E) \leq H^D(E) \leq C\mu(E) \quad \forall \text{ Borel } E \subseteq X.$$

In particular, if the MM space triple (X, d, μ) is regular of dimension D, then so is (X, d, H^D).

Symbolic Cantor sets

Symbolic Cantor set

Let F be a finite set with $k \geq 2$ elements. Then $F^\infty = \{\{x_i\}_{i=1}^\infty : x_i \in F\}$ is the k-Cantor set.

Define the valuation $L(x, y)$, $x = \{x_i\}_{i=1}^\infty$, $y = \{y_i\}_{i=1}^\infty$, by

$$L(x, y) = \ell,$$

where $x_i = y_i \ \forall i \leq \ell, \ x_{\ell+1} \neq y_{\ell+1}$.

Symbolic Cantor sets

Let $a \in (0,1)$. Then

$$d_a(x,y) = a^{L(x,y)}$$

is an ultrametric.

Place the natural probability measure $\mu = \prod_{i=1}^{\infty} \nu, \quad \nu(j) = 1/k$ for $j \in F$.

Then F^{∞} is regular of dimension $D = \dfrac{\log k}{\log a^{-1}}$.

Symbolic Cantor sets

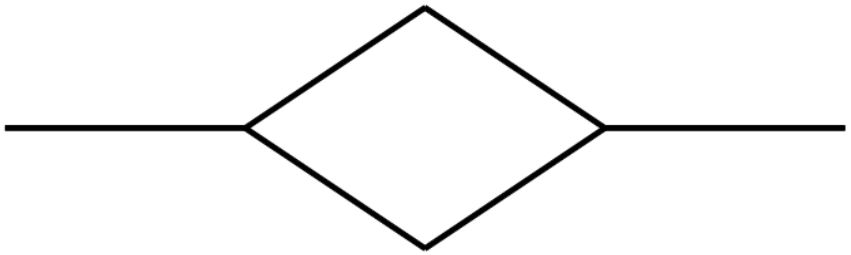
The Cantor set is generated by successive removals of middle thirds.

Laakso graph

Let $X_0 = [0,1]$. For $i > 0$, define X_i by replacing each edge of X_{i-1} by a $4^{-(i-1)}$ scaled copy of Γ (below). Then $\{X_i\}_{i=0}^{\infty}$ forms an inverse system

$$X_0 \xleftarrow{\pi_0} \dots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \dots,$$

where $\pi_{i-1} : X_i \to X_{i-1}$ collapses the copies of Γ at the i-th level.



Graph of Γ

Laakso graph

Then the inverse limit X_∞ is the **Laakso graph**, with metric

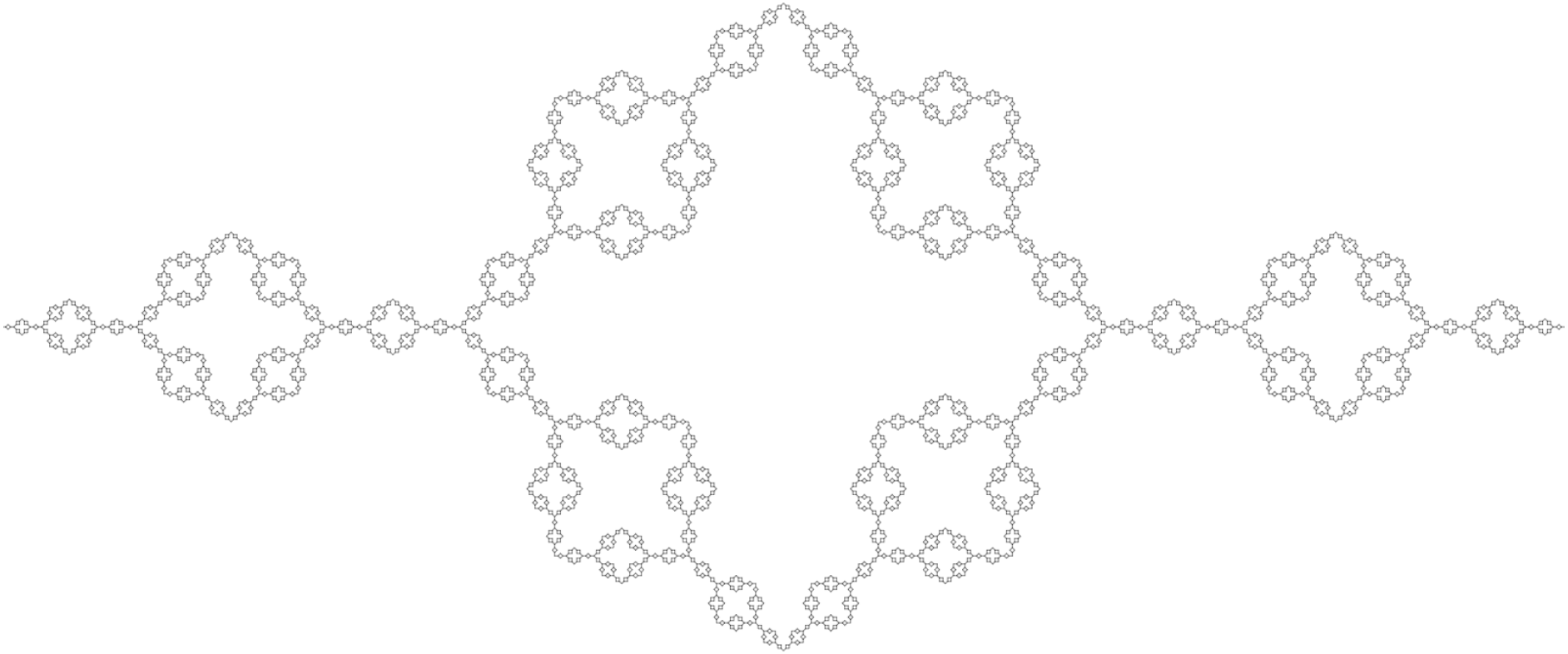
$$d_\infty(x, x') = \lim_{i \to \infty} d_{X_i}(\pi_i^\infty(x), \pi_i^\infty(x')),$$

where X_∞ is the Gromov-Hausdorff limit of $\{X_i\}$ and $\pi_i^\infty : X_\infty \to X_i$ is the canonical projection.

The Laakso graph is a compact, and hence complete, metric space. This guarantees the existence of a doubling measure that makes the Laakso graph an MM space.

Of similar construction is the **Laakso space**, which is regular of dimension $D = 1 + \dfrac{\log 2}{\log 3}$.

Laakso graph



Heisenberg Group

Heisenberg Group

We define the **n-dimensional Heisenberg group** in its algebra representation $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with group multiplication given by

$$(z,t)(z',t') = (z+z', t+t' - \frac{1}{2}\mathrm{Im}\sum_{j=1}^{n} z_j \overline{z_j'}).$$

There is a natural dilation action $\partial_r(z,t) = (rz, r^2 t)$, $r > 0$, that gives rise to the homogeneous norm $\|(z,t)\| = (\sum_{j=1}^{n} |z_j|^4 + t^2)^{\frac{1}{4}}$, with the properties $\|\partial_r(z,t)\| = r\|(z,t)\|$ and $\|x^{-1}\| = \|x\|$.

Heisenberg Group

This norm defines a metric $d(x, y) = \|x^{-1}y\|$.

The Haar measure given by Lebesgue measure under the exponential map gives

$$\mu(B(x, r)) = r^{2n+2}\mu(B(x, 1)).$$

Thus the Heisenberg group is regular of dimension $D = 2n + 2$, while its topological dimension is $T = 2n + 1$.

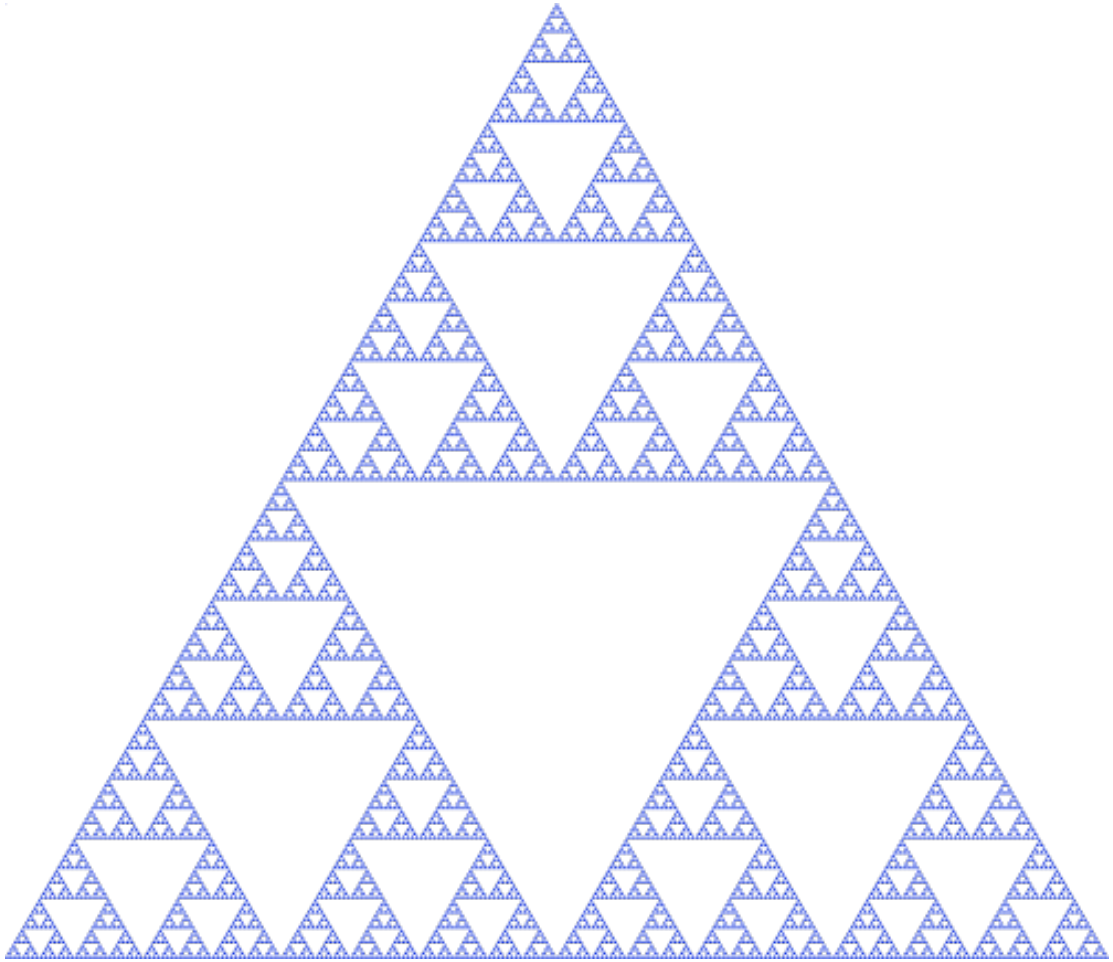
Fractals and Harmonic embeddings

Many self-similar fractals in Euclidean space can be thought of as MM or Ahlfors regular spaces.

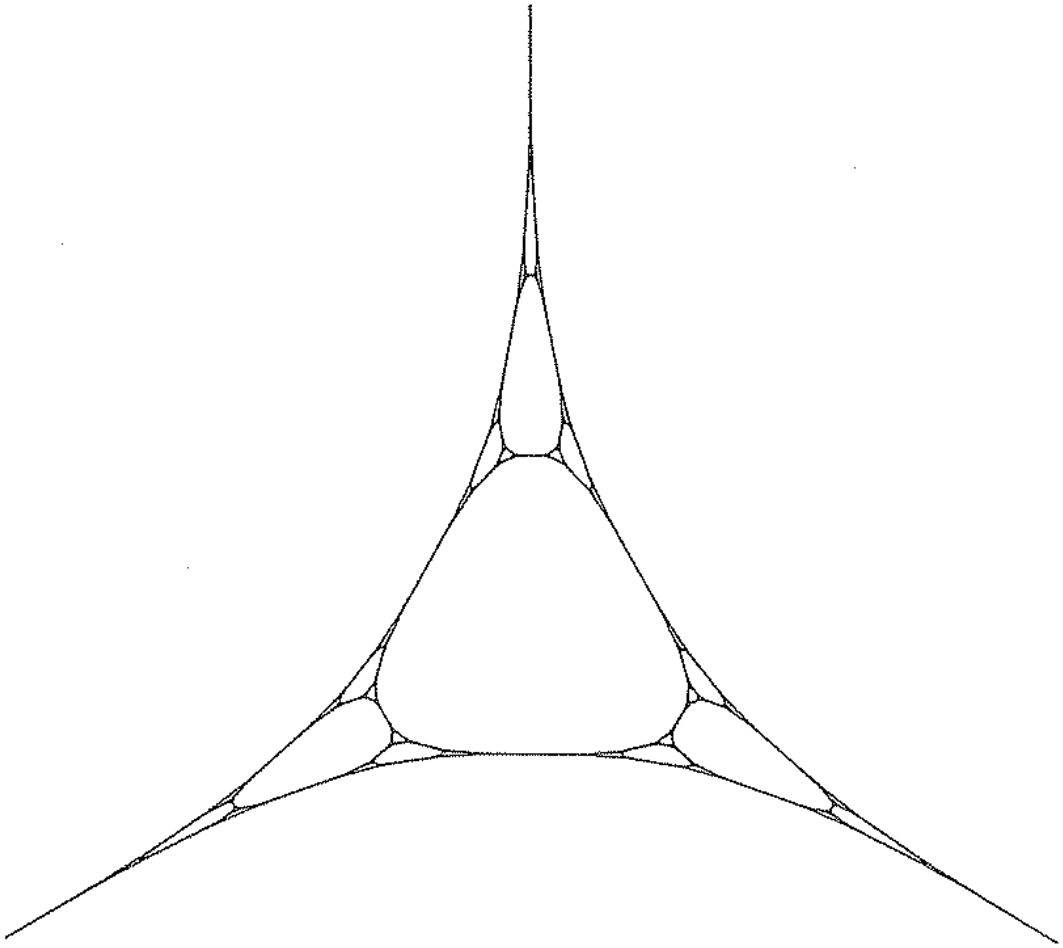
Using key work of Kusuoka, Kigami showed that the Sierpiński gasket could be embedded in $\mathbb{R}^2$ by a certain harmonic map. He also showed the resulting **harmonic Sierpiński gasket** can be viewed as a measurable Riemannian manifold with the intrinsic geodesic metric induced by the Euclidean structure of $\mathbb{R}^2$.

More recently, Kajino proved that the harmonic gasket is an Ahlfors regular space under the associated Hausdorff measure.

Sierpiński Gasket



Harmonic Sierpiński Gasket



Weighted $\mathbb{R}^N$

Weighted $\mathbb{R}^N$

We define **weighted $\mathbb{R}^N$ space** as the triple $(\mathbb{R}^N, d, \mu)$, where d is the standard Euclidean metric and μ is the measure defined as

$$\mathrm{d}\mu = |x|^{\alpha}\mathrm{d}x, \quad \alpha > -N, \ \mathrm{d}x \text{ Lebesgue measure.}$$

Then weighted $\mathbb{R}^N$ space is a MM space, but it is not Ahlfors regular.

What defines a fractal?

Two dimensions that capture how volume scales under dilations or contractions: the Hausdorff dimension and the Minkowski dimension.

Minkowski dimension

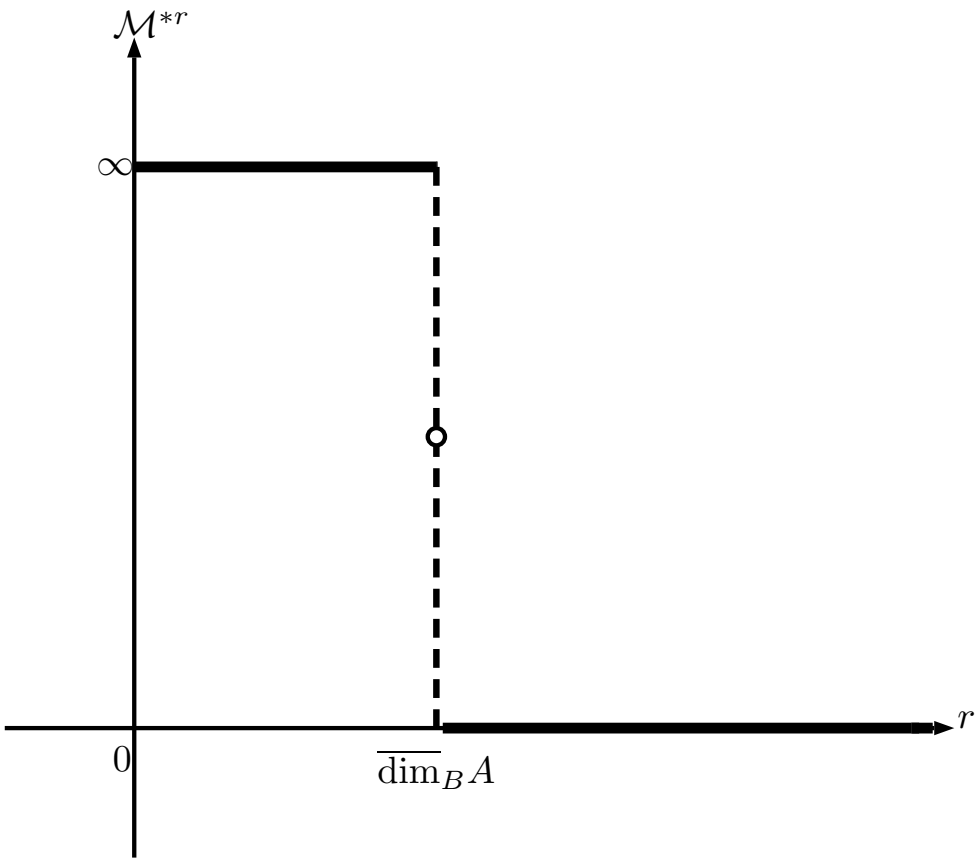
Given $A \subset \mathbb{R}^N$ bounded, define the t-**neighborhood of** A by $A_t := \{x \in \mathbb{R}^N : d(x, A) < t\}$. Then the **r-dimensional Minkowski upper content** is defined as

$$\mathcal{M}^{*r} = \limsup_{t \to 0} \frac{|A_t|}{t^{N-r}}.$$

Define the **upper Minkowski dimension** by

$$\begin{aligned} \overline{\dim}_B A &= \inf\{r \in \mathbb{R} : \mathcal{M}^{*r}(A) = 0\} \\ &= \sup\{r \in \mathbb{R} : \mathcal{M}^{*r}(A) = \infty\}. \end{aligned}$$

What defines a fractal?



An example of the r-dimensional Minkowski upper content of a set A with Minkowski dimension $\overline{\dim}_B A$.

What defines a fractal?

We define the **r-dimensional lower Minkowski content** $\mathcal{M}_*^r$ and **lower Minkowski dimension** $\underline{\dim}_B(A)$ analogously.

Minkowski dimension cont.

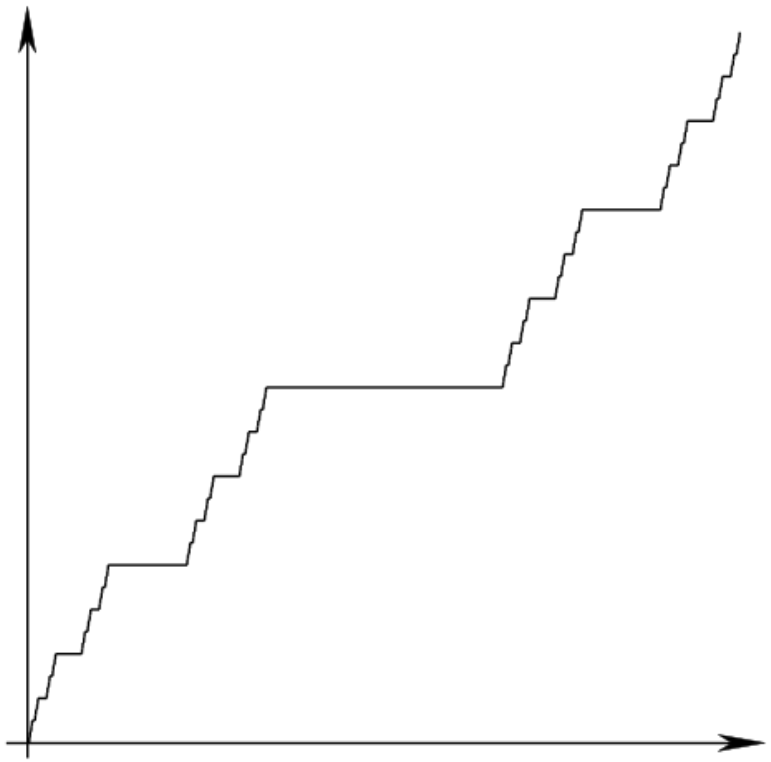
Given a set $A \subset \mathbb{R}$, if

$$\overline{\dim}_B A = \underline{\dim}_B A,$$

then we call their common value $\dim_B A$ the **Minkowski dimension** of A.

However, it is not enough to say that non-integer scaling dimensions or scaling dimensions larger than topological dimension defines a fractal.

What defines a fractal?



The Devil's staircase, which has Minkowski, Hausdorff and topological dimensions $D = 1$.

What defines a fractal?

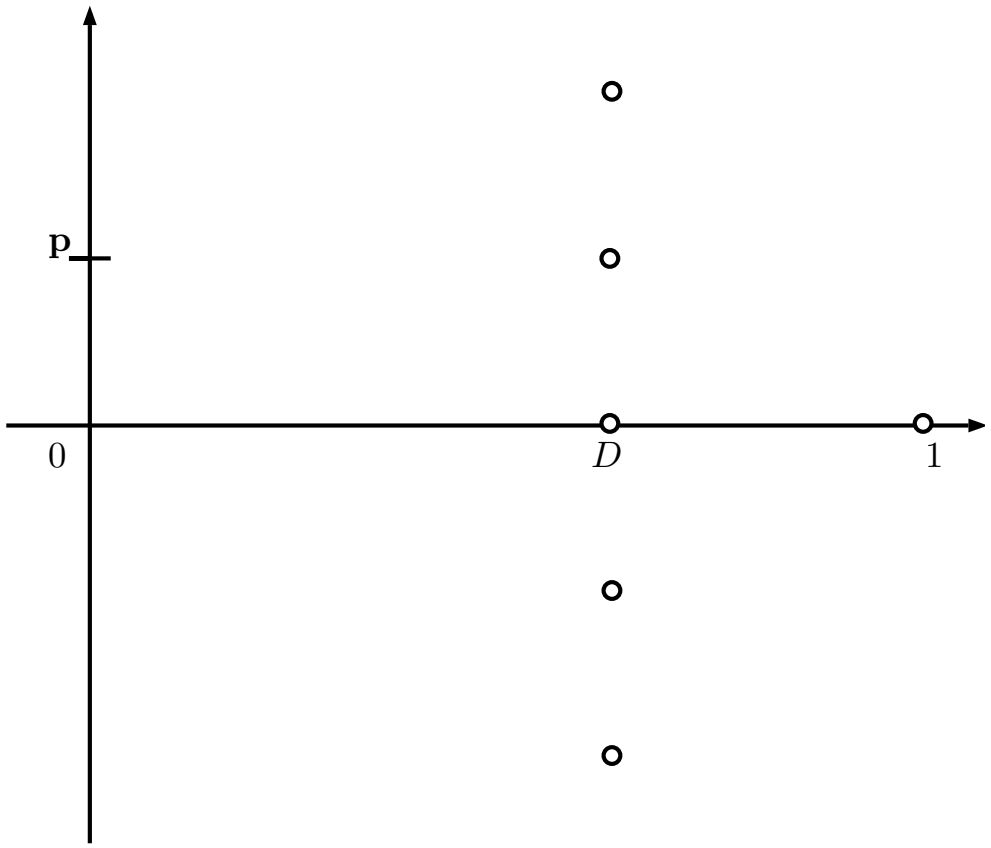
However, as done in [FGCD], we find that by studying the volume of the ϵ-neigborhoods of the Devil's staircase, it is approximated by

$$V(\epsilon) \approx 2\epsilon^{2-1} + \frac{4-\pi}{8\log 3} \sum_{n=-\infty}^{\infty} \frac{\epsilon^{(2-D-in\mathbf{p})}}{(D+in\mathbf{p})(1-D-in\mathbf{p})},$$

where $D = \log_3 2$ and $\mathbf{p} = 2\pi/\log 3$.

Viewing the exponents of ϵ as codimensions, we see that the dimensions associated to the geometry of the Devil's staircase are 1 and $D + in\mathbf{p}$ for any $n \in \mathbb{Z}$. In particular, there are complex values associated to these dimensions. We call the entire set of dimensions the **complex dimensions** of the Devil's staircase.

What defines a fractal?



The complex dimensions of the Devil's staircase, with $D = \log_3 2$ and $\mathbf{p} = 2\pi / \log 3$.

Fractal Strings

The theory of complex dimensions in $\mathbb{R}$ was developed through the use of fractal strings (one-dimensional fractal drums) in [FGCD].

Fractal String

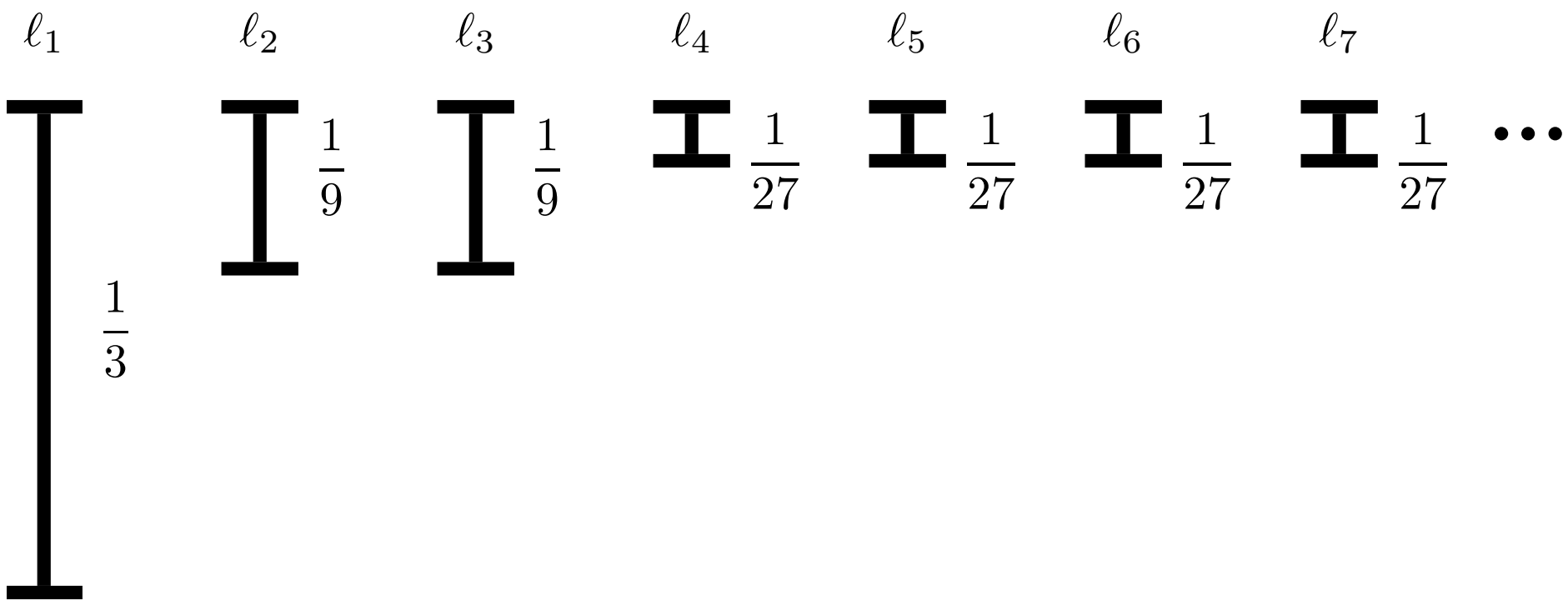
A **fractal string** is a bounded open subset of the real line; i.e. it is a disjoint union of open intervals (the boundary of which may be fractal). The lengths of these open intervals forms a non-increasing sequence

$$\mathcal{L} = \ell_1, \ell_2, \ell_3, \ldots$$

For example, we define the **Cantor string**, CS, as the complement of the ternary Cantor set in $[0, 1]$. Thus in terms of the lengths of the disjoint open intervals,

$$\mathrm{CS} = \left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \ldots \right\}$$

Fractal Strings



The Cantor string represented as a fractal harp.

Fractal Strings

Geometric Zeta Function

Given a fractal string $\mathcal{L}$, we define its complex valued **geometric zeta function**, $\zeta_{\mathcal{L}}$ as

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s.$$

This function is holomorphic on Re $s > D$ =Minkowski dimension of $\partial\mathcal{L}$. Moreover, this half-plane is optimal in the sense that D is the abscissa of absolute convergence of $\zeta_{\mathcal{L}}$.

We define the **complex dimensions** as the poles of the meromorphic continuation.

Fractal Strings

Thus, the Cantor String has geometric zeta function

$$\zeta_{CS}(s) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}}.$$

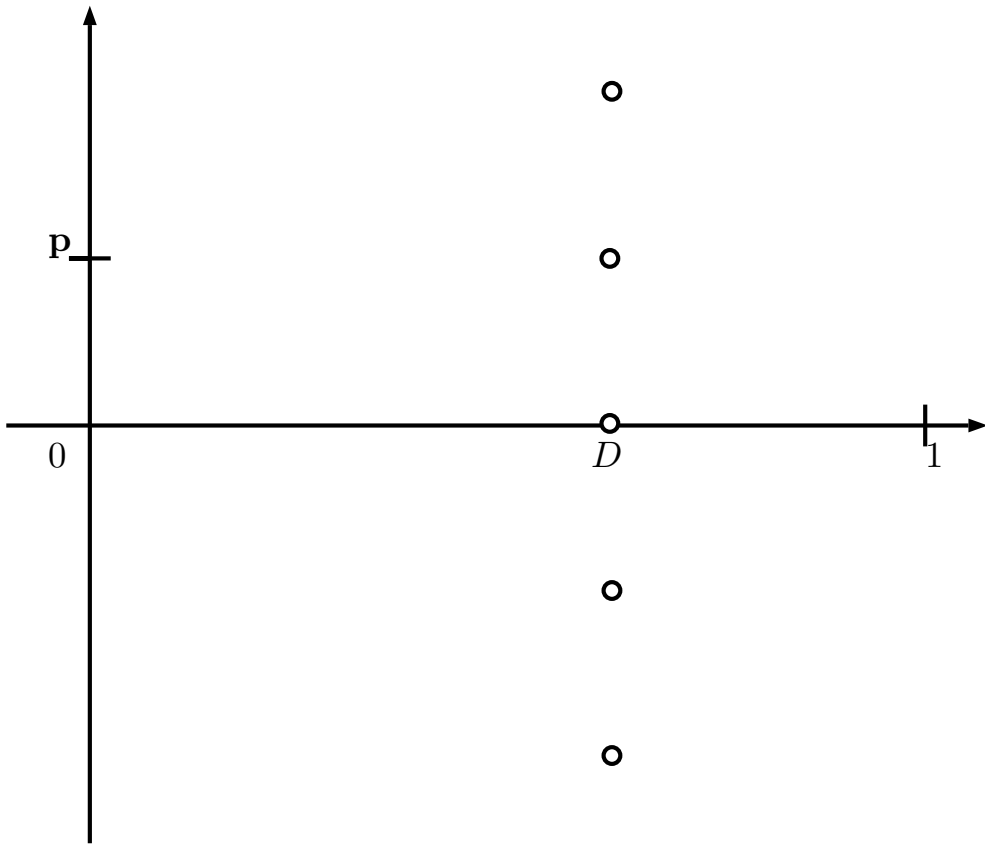
This is absolutely convergent and holomorphic for Re $s > \log_3 2 = D$. It has a meromorphic continuation to all of $\mathbb{C}$ given by

$$\zeta_{CS}(s) = \frac{1}{3^s} \cdot \frac{1}{1 - 2 \cdot 3^{-s}},$$

with complex dimensions equal to the set of poles

$$\left\{ D + in\frac{2\pi}{\log 3} : n \in \mathbb{Z} \right\}$$

Fractal Strings



The complex dimensions of the ternary Cantor set, with $D = \log_3 2$ and $\mathbf{p} = 2\pi / \log 3$.

Fractal Strings

We find that the Cantor String tube formula (volume of the inner ϵ-tube) is

$$V_{CS}(\epsilon) = \frac{1}{2\log 3} \sum_{n=-\infty}^{\infty} \frac{(2\epsilon)^{(1-D-in\mathbf{p})}}{(D+in\mathbf{p})(1-D-in\mathbf{p})} - 2\epsilon$$

where $\mathbf{p} = 2\pi/\log 3$ and $\frac{1}{2\log 3}$ is the residue of ζ_{CS} at each of the poles.

Fractal Strings

Spectrum and Riemann Hypothesis

Given a Minkowski measurable fractal string $\mathcal{L}$, its frequency (spectral) counting function $N_{\nu,\mathcal{L}}(\mu)$ admits a monotonic asymptotic second term of the form $-c_D \mathcal{M} \mu^{D/2}$, where $D \in (0,1)$.

The constant c_D depends only on D and is directly propotional to $-\zeta(D)$, the Riemann zeta function.

This result was obtained by Lapidus and Pomerance, thereby establishing a connection between the direct spectral problem, Minkowski measurability, and the Riemann zeta function.

Fractal Strings

Inverse Spectral Problem

Given that $\mathcal{L}$ is a fractal string for which the spectral counting function $N_{\nu,\mathcal{L}}(\mu)$ admits a monotonic asymptotic second term proportional to $\mu^{D/2}$ (as $\mu \to \infty$), does it follow that $\mathcal{L}$ is Minkowski measurable?

It has been shown by Lapidus and Maier that the inverse spectral problem is intimately connected with the location of the critical zeros of $\zeta(s)$.

Theorem [LaMa]

The inverse spectral problem has an affirmative answer for all $D \in (0, 1)$, with $D \neq 1/2$, if and only if the Riemann hypothesis is true.

Distance Zeta Function

In order to generalize the theory to $\mathbb{R}^N$, the distance zeta function was introduced in [FZF].

Distance Zeta Function

Given a bounded set $A \subset \mathbb{R}^N$, $\delta > 0$, we define the **distance zeta function** as

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x,$$

for $s \in \mathbb{C}$ with Re s sufficiently large, and the integral taken in the Lebesgue sense.

It will turn out that for the properties of this function we wish to study, the value of δ will not matter.

Distance Zeta Function

Theorem [FZF]

Given a bounded set $A \subset \mathbb{R}^N$, $\delta > 0$, we define the distance zeta function

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x.$$

Then ζ_A is holomorphic in the half-plane $\{\mathrm{Re}\ s > \overline{\dim}_B(A)\}$ with

$$\zeta_A'(s) = \int_{A_\delta} d(x, A)^{s-N} \log d(x, A) \mathrm{d}\mu.$$

We have that $\overline{\dim}_B(A)$ is optimal, in the sense that it is the abscissa of Lebesgue convergence of ζ_A.

Further, if $D = \dim_B(A)$ exists and $\mathcal{M}_*^D > 0$, then $\zeta_A(s) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right.

Distance Zeta Function

If ζ_A can be meromorphically extended, then we call the poles of such an extension the **complex dimensions** of the set A.

As we shall see, these poles will differ slightly compared to the geometric zeta function for fractal strings. It is currently unknown if this is perhaps due to the geometric realization of the fractal strings used.

However, the **principal complex dimensions**, the poles above the critical line $\{\text{Re } s = D\}$ where D is the abscissa of holomorphic convergence, will coincide.

Distance Zeta Function

Given any nontrivial fractal string, $\mathcal{L}$, define $A := \{a_k : k \geq 1\}$, where $a_k := \sum_{j \geq k} l_j$. Then we have that

$$\zeta_A(s) = u(s)\zeta_{\mathcal{L}}(s) + v(s)$$

where u, v are both holomorphic functions in the right half-plane $\{s \in \mathbb{C} : \text{Re } s > 0\}$.

Further, the set of poles of the meromorphic extensions of ζ_A and $\zeta_{\mathcal{L}}$ coincide to the right of any open half-plane $\{\text{Re } s > c\}$ for $c > 0$.
In particular, the complex dimensions coincide to the right of $\{\text{Re } s = 0\}$.

Tube Zeta Function

The distance zeta function can be written in the following way: for any Re $s > \overline{\dim}_B(A)$,

$$\int_{A_\delta} d(x,A)^{s-N}\mathrm{d}x = \delta^{s-N}|A_\delta| + (N-s)\int_0^\delta t^{s-N-1}|A_t|\mathrm{d}t.$$

Tube Zeta Function

Let $\delta > 0$, A a bounded set in $\mathbb{R}^N$. Then the **tube zeta function** of A, $\tilde{\zeta}_A$, is defined as

$$\tilde{\zeta}_A(s) = \int_0^\delta t^{s-N-1}|A_t|\mathrm{d}t$$

Tube Zeta Function

Provided that $\overline{\dim}_B(A) < N$, $\tilde{\zeta}_A$ shares important properties with ζ_A.

Theorem [FZF]

Assume A is a bounded subset of $\mathbb{R}^N$ with $\overline{\dim}_B(A) < N$. Then $\tilde{\zeta}_A$ is holomorphic in the half-plane $\{\text{Re } s > \overline{\dim}_B(A)\}$, where $\overline{\dim}_B(A)$ is the abscissa of Lebesgue convergence.

Moreover, $\tilde{\zeta}_A$ and ζ_A will share the same domain of meromorphic extension (if it exists) with the same poles and order.

In particular, the complex dimensions are the same.

Residues and Minkowski Content

Theorem [FZF]

Assume that the bounded set $A \subset \mathbb{R}^N$ is Minkowski nondegenerate (that is, $\dim_B A = D$ and $0 < \mathcal{M}_*^D(A) \leq \mathcal{M}^{*D}(A) < \infty$,) and $D < N$. If $\zeta_A(s)$ can be meromorphically extended to a neighborhood of $s = D$, then D is necessarily a simple pole of $\zeta_A(s)$ and

$$(N-D)\mathcal{M}_*^D(A) \leq \mathrm{res}(\zeta_A(\cdot), D) = (N-D)\mathrm{res}(\tilde{\zeta}_A(\cdot), D) \leq (N-D)\mathcal{M}^{*D}(A).$$

Residues and Minkowski Content

Let A be the ternary Cantor set, and let $\delta \geq 1/6$. Then we obtain

$$\zeta_A(s) = 2^{1-s}s^{-1}\frac{3^{-s}}{1-2\cdot 3^{-s}} + 2\delta^s s^{-1}$$

Its residue at $D(A) = \log_3 2$ is equal to

$$\operatorname{res}(\zeta_A(\cdot), D(A)) = \frac{2}{\log 2}\left(\frac{1}{6}\right)^{\log_3 2 - 1} = \frac{1}{2^{\log_3 2}\log 2},$$

while its upper and lower Minkowski contents are given by

$$\mathcal{M}_*^{\log_3 2}(A) = \frac{\log 9}{\log 3/2}\left(\frac{\log 3/2}{\log 4}\right)^{\log_3 2}, \quad \mathcal{M}^{*\log_3 2}(A) = 2^{2-\log_3 2}.$$

Residues and Minkowski Content

More generally, at each of the poles on the critical line $\{\mathrm{Re}s = \log_3 2\}$, $s_k := \log_3 2 + k\mathbf{p}i,\ k \in \mathbb{Z}$,with $\mathbf{p} := \dfrac{2\pi}{\log 3}$, we have

$$\mathrm{res}(\zeta_A(\cdot), s_k) = \frac{\log_3 2}{s_k 2^{k\mathbf{p}i}} \mathrm{res}(\zeta_A(\cdot), D(A)).$$

In particular, it is noteworthy that these residues tend to zero as $k \to \pm\infty$.

Sierpiński carpet

In parallel to the fractal string case, we will call $V(t) := |A_t|$ the volume of the t-tube neighborhood.

Let A be the standard Sierpiński carpet in the plane. Then

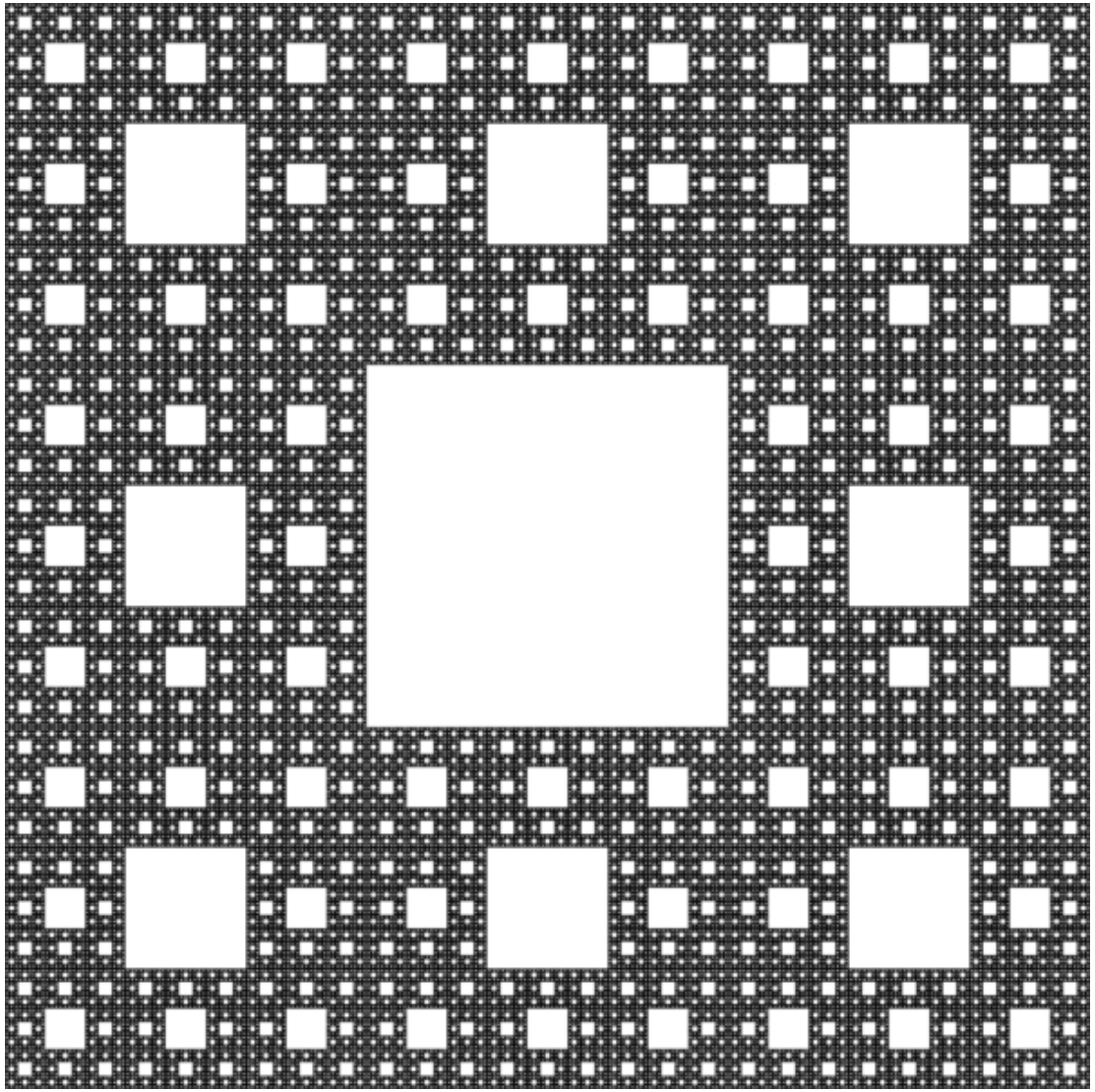
$$|A_t| = t^{2-D}(G(\log t^{-1}) + O(t^{D-1}))$$

as $t \to 0$, where $D = \log_3 8$ and G is a nonconstant periodic function with period $T = \log 3$.

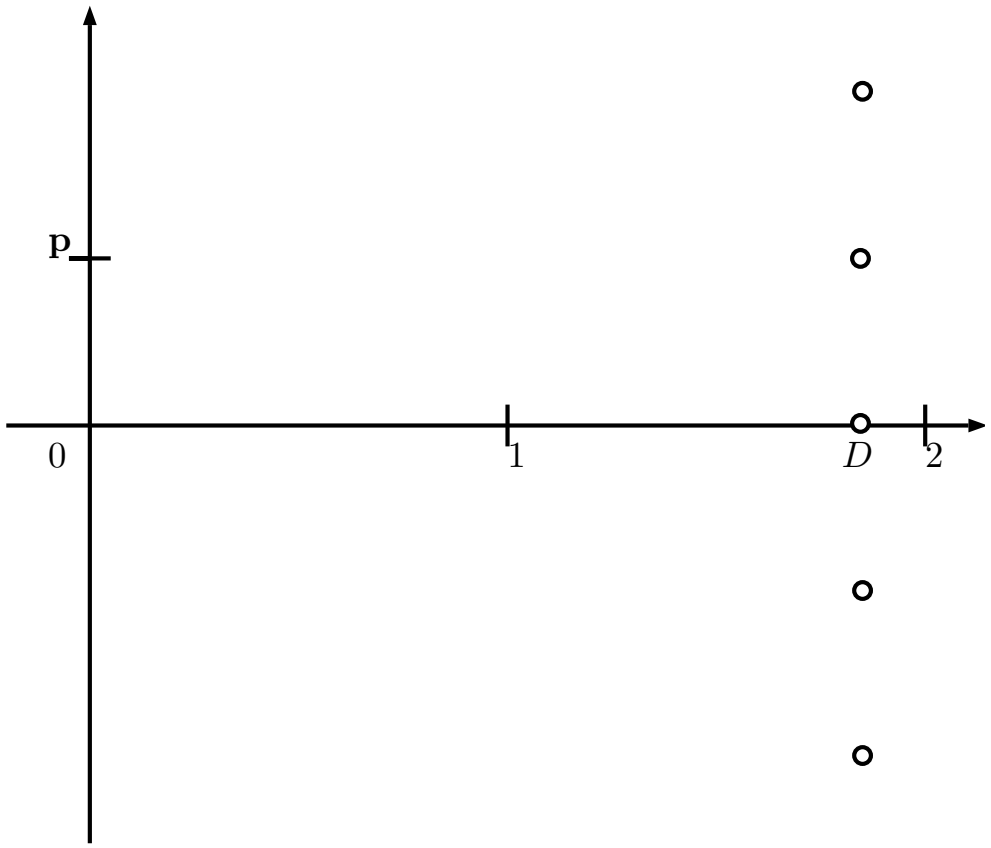
By direct computation we can find that both zeta functions have a meromorphic extension to all of $\mathbb{C}$, and the set of complex dimensions of A are simple poles given by

$$\dim_{\mathbb{C}} A = \left\{ D + \frac{2\pi}{\log 3} ki : k \in \mathbb{Z} \right\}.$$

Sierpiński carpet



Sierpiński carpet



The complex dimensions of the Sierpiński carpet, with $D = \log_3 8$ and $\mathbf{p} = 2\pi / \log 3$.

Sphere in $\mathbb{R}^N$

Let $B_R(0)$ be the open ball in $\mathbb{R}^n$ with radius R, and let $A = \partial B_R(0)$ be the $(N-1)$-dimensional sphere with radius R. Further, let $c_k = 1 - (-1)^k$ and $\omega_N = |B_1(0)|$.

Fix $\delta < R$. Then $\tilde{\zeta}_A(s)$ meromorphically extends to all $\mathbb{C}$, with representation

$$\tilde{\zeta}_A(s) = \omega_N \sum_{k=0}^{N} c_k R^{n-k} \binom{N}{k} \frac{\delta^{s-N+k}}{s-(N-k)}.$$

As expected, we get $D(A) = N-1$, although (perhaps surprisingly) its set of complex dimensions is

$$\dim_{\mathbb{C}}(A) = \left\{ N-1, N-3, \ldots, N-(2\left\lfloor \frac{N-1}{2} \right\rfloor + 1) \right\}.$$

Fractality

The introduction of complex dimensions has led to a conjectured definition of fractality, first in [FGCD] and then further extended in [FZF], that would not leave out exceptional cases:

Definition

A geometric object is said to be a **fractal subset of** $\mathbb{R}^N$ if its associated fractal zeta function has at least one nonreal complex dimension.

This would include not only classical fractals, but former exceptions such as the Devil's staircase.

Generalized Minkowski Dimension

Minkowski Dimension in MM spaces

Let X be an Ahlfors regular MM space of dimension D_H. Then we define the **r-dimensional upper Minkowski content** by

$$\mathcal{M}^{*r} = \limsup_{t \to 0} \frac{|A_t|}{t^{D_H - r}}.$$

Lower content and Minkowski dimension are then defined analogously to the Euclidean case.

Ahlfors regularity is necessary to insure that the "ambient space" has consistent dimension, or generally that single points have Minkowski dimension 0. See weighted $\mathbb{R}^N$ space at $A = \{0\}$ for a counterexample.

Preliminary Results

Theorem (Lapidus, W.)

If we define $D(A) = \overline{\dim}_B A$, then the distance zeta function,

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s - D_H} \mathrm{d}\mu,$$

is holomorphic in the half plane $\{\mathrm{Re}\ s > D(A)\}$, with

$$\zeta_A'(s) = \int_{A_\delta} d(x, A)^{s - D_H} \log d(x, A) \mathrm{d}\mu.$$

We have that $D(A)$ is optimal, in the sense that it is the abscissa of Lebesgue convergence of ζ_A.

Further, if $D = \dim_B(A)$ exists and $\mathcal{M}_*^D > 0$, then $\zeta_A(s) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right.

Future Work

- Is the analogous tube zeta function well defined and do further results involving these fractal zeta functions (relative fractal zeta functions, spectrums, etc.) generalize to MM spaces?
- Find and study examples of fractal sets (sets with nonreal complex dimensions) in MM spaces.
- Examine what information the complex dimensions give concerning the geometry of such fractal sets, in analogy to the Euclidean case.

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