Parabolic SPDE in metric measure spaces - regularity of the solution

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Main references (joint work):

Potential theoretic background:
Jiaxin Hu, M.Z. : Studia Math. 2005, Potential Anal. 2009
Existence and uniqueness problems in fractional Sobolev-type spaces: Michael Hinz, M.Z.: Semigroups, potential spaces and applications to (S)PDE, Potential Anal. 2012
Hölder regularity: Elena Issoglio, M.Z.: Regularity of the solution to SPDE in metric measure spaces (Preprint)

Related literature:

Elliptic and some parabolic PDE on self-similar sets with respect to fractal Laplace operators (without noise terms): Falconer, Grigoryan, Hu, Khoshnevisan, Strichartz, ...
Abstract parabolic problems with Brownian and fractional Brownian noise: DaPrato/Zabzcyk, Dalang, Grecksch/Anh, Maslowski/Nualart, Tindel/Tudor/Viens, Prévot/Röckner, Röckner/Wang, ...
more general noise, Young integrals, local solution: Gubinelli/Lejay/Tindel

Euclidean versions of the results below (Hinz/Z., JFA 2009):

formal Cauchy problem on bounded smooth domains in $\mathbb{R}^n$

$$\frac{\partial u}{\partial t}(t,x) = -Au(t,x) + F(u(t,x)) + < G(u(t,x)), \frac{\partial}{\partial t}\nabla Z(t,x) >, \; t \in (0,t_0],$$

A an elliptic operator, $\frac{\partial}{\partial t}\nabla Z$ fractional space-time noise, initial value and Dirichlet boundary conditions, pathwise approach

1. Extension to non-linear spaces

$[X, \mu]$ a certain locally finite metric measure space

consider the formal Cauchy problem on $[0, t_0] \times X$

$$\frac{\partial u}{\partial t} = -A^\theta u + F(u) + G(u) \cdot \dot{z}^*, \quad t \in (0, t_0], \qquad (1)$$

with initial condition $u(0, x) = f(x)$, where

- $-A$ is the generator of a strongly continuous Markovian symmetric semigroup $(T_t)_{t \geq 0}$ on $L_2(\mu)$ satisfying sub-Gaussian heat kernel estimates
  (e.g. the Dirichlet Laplacian in the above models)
- A^θ, $\theta \leq 1$, is a fractional power of A
- F and G are sufficiently regular functions on $\mathbb{R}$
- z^* is a random element in $C^{1-\alpha}\big([0, t_0], H_q^{\theta\beta}(\mu)^*\big)$
- $f \in H_2^{2\gamma+\theta\delta+\varepsilon}(\mu)$

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Aim: pathwise mild function solution $u \in C^{\gamma}\big([0,t_0], H_2^{\theta\delta}(\mu)\big)$

for $\theta = 1$ mild solution defined by

$$u(t) = T_t f + \int_0^t T_{t-s} F(u(s))\, ds + \int_0^t T_{t-s} G(u(s)) dz^*(s) \qquad (2)$$

where $u(t) := u(t, \cdot)$ and the last formal integral to be determined,
(for $\theta < 1$ use the subordinated semigroup T^{θ} with generator $-A^{\theta}$ instead of T_t)

Main ideas:

- the spatial smoothness is measured in terms of potential spaces $H_2^{\sigma}(\mu)$ generated by the semigroup, and the latter lifts certain dual spaces to function spaces,
- the paraproduct is introduced by duality relations
- the time integral is realized by means of *Banach space valued fractional calculus*

our approach is independent of series expansions

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2. Bessel potential spaces associated with semigroups on (metric) measure spaces

Main assumptions:

- (X, d, μ) locally compact metric measure space, μ Radon measure, $X = \mathrm{supp}\mu$ admitting a
- strongly continuous Markov semigroup $(T_t)_{t\geq 0}$ on $L_2(\mu)$
- $$T_t = e^{-At}, \quad -A \text{ infinitesimal generator},$$
- T_t has transition densities $p_t(x,y)$ possessing generalized sub-Gaussian estimates with parameter β

$$t^{-\frac{d_f}{w}}\Phi_1(t^{-\frac{1}{w}}d(x,y)) \leq p_t(x,y) \leq t^{-\frac{d_f}{w}}\Phi_2(t^{-\frac{1}{w}}d(x,y))$$

if $0 < t < R_0$, and $p_t(x,y) \leq e^{\omega t}$ if $t \geq R_0$, for any $x, y \in X$ and some constants $R_0 > 0$ and $0 < \omega < 1$, with bounded decreasing functions Φ_i on $(0,\infty)$, Φ_2 such that

$$\int_0^\infty s^{d_f+\beta/2-1}\Phi_2(s)\mathrm{d}s < \infty$$

.

Fractional powers of A are determined by:

$$A^{\alpha}u = \mathrm{const}(\alpha, l) \int_0^\infty t^{-\alpha-1}(I - T_t)^l u \, dt$$

for $l > \alpha > 0$ and

$$A^{-\alpha}\varphi = \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} T_t \varphi \, dt$$

for $\alpha > 0$ and all $\varphi \in L_2(\mu)$ if 0 is in the resolvent of A

define for $\sigma \geq 0$ and some $\omega > 0$:
Bessel potential operators: (take $e^{-\omega t}T_t$ instead of T_t)

$$J^{\sigma} := (\omega I + A)^{-\sigma/2}$$

Bessel potential spaces: $H_2^{\sigma}(\mu) := J^{\sigma}(L_2(\mu))$ with norm

$$||u||_{H_2^{\sigma}(\mu)} := ||(\omega I + A)^{\sigma/2}(u)||_{L_2(\mu)} \sim ||u||_{L_2(\mu)} + ||A^{\sigma/2}(u)||_{L_2(\mu)}$$

(resp. for all $p > 1$, $H_p^{\sigma}(\mu) := J^{\sigma}(L_p(\mu))$ with norm

$$||u||_{H_p^{\sigma}(\mu)} := ||(\omega I + A)^{\sigma/2}(u)||_{L_p(\mu)} \sim ||u||_{L_p(\mu)} + ||A^{\sigma/2}(u)||_{L_p(\mu)} \,)$$

we also consider the spaces

$$H^\sigma_{2,\infty}(\mu) := H^\sigma_2(\mu) \cap L_\infty(\mu)$$

with norm

$$||u||_{H^\sigma_{2,\infty}}(\mu) := ||u||_{H^\sigma_A(\mu)} + ||u||_{L_\infty(\mu)}$$

Dual spaces:

$$H^\sigma_p(\mu)^* =: H^{-\sigma}_{p'}(\mu)$$

with $\sigma \geq 0$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.

By duality the operators T_t can be extended to the dual spaces: If $w \in H^{-\sigma}(\mu)$ then $T_t w$ is the element of $L_2(\mu)$ determined by the scalar products

$$(v, T_t w) := (T_t v, w)\,, \quad v \in L_2(\mu)\,.$$

(Note that $|(v, T_t w)| = |(T_t v, w)| \leq ||T_t v||_{H^\sigma_p(\mu)} ||w||_{H^{-\sigma}_{p'}(\mu)}$.)

Mapping properties of T_t in the above spaces are implied.

(for L_∞-properties the ultracontractivity is used, for paraproducts the full heat kernel estimates - fulfilled for many examples, classical and fractal cases)

Application to parabolic SPDE on fractals:

- fractal Laplacians: Lindstrøm, Barlow, Bass, Kusuoka, Strichartz, Kigami (and many others)
- these fractals are special metric measure spaces fulfilling the above assumptions: the Laplacians generate semigroups with sub-Gaussian heat kernel estimates

3. Rigorous definition and solution of the stochastic partial (pseudo) differential equation

general situation as above: (T_t) with generator $-A$ (or T_t^θ with generator $-A^\theta$ instead);
recall that u is a mild solution of the Cauchy problem (1) if

$$u(t) = T_t f + \int_0^t T_{t-s} F(u(s))\, ds + \int_0^t T_{t-s} G(u(s))\, dz^*(ds)$$

rewrite the last formal integral as

$$\int_0^t U_t(s)\, dz^*(s)$$

where

$$U_t(s)(w) := T_{t-s}(G(u(s)) \cdot w)$$

is for fixed sufficiently nice u with values in $H^\delta_{2,\infty}(\mu)$ and $q := \frac{d_S}{\delta}$ shown to be a mapping

$$U_t : [0, t_0] \to L\left(H_q^{-\beta}(\mu), H^\delta_{2,\infty}(\mu)\right)$$

with fractional order of smoothness α' slightly greater than α,

by assumption $z^* \in C^{1-\alpha}\left([0, t_0], H_q^{-\beta}(\mu)\right)$, and consequently, has fractional order of smoothness $1 - \alpha' < 1 - \alpha$, so that we can define

$$\int_0^t U_t(s)\, dz^*(s) := \int_0^t D^{\alpha'}_{0+} U_t(s) \left(D^{1-\alpha'}_{t-} z^*_t(s)\right) ds$$

for left and right sided fractional derivatives $D^{\alpha'}_{0+}$ and $D^{1-\alpha'}_{t-}$ (and $z^*_t := z^* - z^*(t)$).

If the noise coefficient functions are **linear**, the L_∞-norms can be omitted. This leads to solutions for all spectral dimensions: Let $0 < \theta \leq 1$.

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Theorem. (i) For linear coefficient functions F and G:

If $z^* \in C^{1-\alpha}([0,t_0], H_q^{-\theta\beta}(\mu))$, $q = \frac{d_S}{\delta}$, $f \in H_2^{2\gamma+\theta\delta}(\mu)$, $0 < \alpha, \beta, \gamma, \delta < 1$, $\alpha < \gamma$, $\beta < \delta < \frac{d_S}{2\theta}$ and

$$\gamma + \frac{\delta}{2} < 1 - \alpha - \frac{\beta}{2}\,,$$

then problem (1) has a unique mild solution $u \in W^\gamma\left([0,t_0], H_2^{\theta\delta}(\mu)\right)$. This solution is also an element of $C^\gamma\left([0,t_0], H_2^{\theta\delta}(\mu)\right)$.

(ii) For nonlinear F and G: the same for $W^\gamma\left([0,t_0], H_{2,\infty}^{\theta\delta}(\mu)\right)$ and $C^\gamma([0,t_0], H_2^{\theta\delta}(\mu))$, if the last inequality in (i) is replaced by

$$\gamma + \frac{d_s}{4\theta} < 1 - \alpha - \frac{\beta}{2}$$

$$\left(||u||_{W^\gamma([0,t_0],H)} := \sup_{t\in[0,t_0]} \left(||u(t)||_H + \int_0^t \frac{||u(t)-u(s)||_H}{|t-s|^{1+\gamma}} ds\right)\right)$$

Standard example for $\mathbf{z}^*$: let $\{e_i\}_{i\in\mathbb{N}}$ be a complete orthonormal system of eigenfunctions of A in $L_2(\mu)$ and λ_i be the corresponding eigenvalues, $\{B_i^H(t)\}_{i\in\mathbb{N}}$ are i.i.d fractional Brownian motions in $\mathbb{R}$ with Hurst exponent $0 < H < 1$, and take for z^* the formal series

$$b^H(t) := \sum_{i=1}^{\infty} B_i^H(t)\, q_i\, e_i \quad \text{with} \quad \sum_{i=1}^{\infty} q_i^2\, \lambda_i^{-2\beta_*} < \infty\,,$$

for real coefficients q_i and a parameter $\beta_* \in (0,1)$, then we have a.s.

$$z^* = b^H \in C^{1-\alpha}\left([0,t_0], H_q^{-\beta}(\mu)\right)$$

for any $0 < 1-\alpha < H$ and any
(a) $\beta_* \le \beta < 1$ and $1 < q \le 2$, or
(b) $\beta_* < \beta < 1$ and $q > 2$, provided the eigenfunctions satisfy
$|e_i(x) - e_i(y)| \le c\lambda_i^{\varepsilon/2} d(x,y)^{\varepsilon}$ for some $\varepsilon > 0$
(sufficient conditions: $d_S < 2$ or Hölder continuity of the heat kernel)

(convergence of the series in these spaces)