Singularity of power dissipation in fractal AC circuits

Patricia Alonso Ruiz

University of Connecticut

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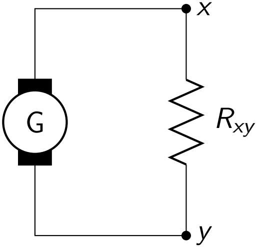
Passive linear networks. Resistors



Ohm's law

$$V_{xy} = I_{xy} R_{xy}.$$

Kirchoff's voltage law

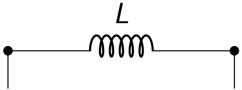
$$V_{xy} = v(x) - v(y),$$

$(v(x), v(y)) \in \mathbb{R}^2$ potential function.

Passive linear networks. Inductors and capacitors

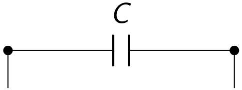
Time-dependent voltage $V(t)$ and current $I(t)$ functions.

Inductor



$$V(t) = L\frac{d}{dt}I(t).$$

Capacitor



$$I(t) = C\frac{d}{dt}V(t).$$

Frequency domain. Impedances

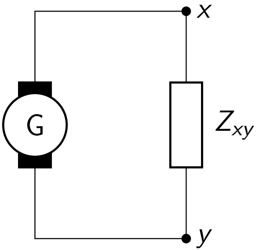
Fourier transform: $\widehat{V}(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} v(t)e^{-i\omega t}dt.$

Inductor: $\widehat{V}(\omega) = i\omega L\,\widehat{I}(\omega) =: Z_L\,\widehat{I}(\omega),$

Capacitor: $\widehat{V}(\omega) = \dfrac{1}{i\omega C}\widehat{I}(\omega) =: Z_C\,\widehat{I}(\omega),$

Resistor: $\widehat{V}(\omega) = R\widehat{I}(\omega) =: Z_R\,\widehat{I}(\omega).$

Ohm's law revisited



Ohm's law (complex-valued)

$$V_{xy}(\omega) = I_{xy}(\omega) Z_{xy}(\omega).$$

Kirchoff's voltage law

$$V_{xy}(\omega) = v(\omega, x) - v(\omega, y),$$

$(v(\omega, x), v(\omega, y)) \in \mathbb{C}^2$ potential function.

Electromotive force

From now on: frequency ω is fixed, φ phase shift.

$$V_{xy}(t) = |V_{xy}|e^{i\omega t}, \quad I_{xy}(t) = |I_{xy}|e^{i(\omega t - \varphi)}, \quad Z_{xy} = |Z_{xy}|e^{i\varphi}.$$

Electromotive force

$$\mathrm{emf}_{xy}(t) = I_{xy}(t)Z_{xy} = |I_{xy}||Z_{xy}|e^{i\omega t},$$

Power dissipation

Average energy loss

$$\frac{1}{T}\int_0^T \Re(\mathrm{emf}_{xy}(t))\Re(I_{xy}(t))\,dt = \cdots = \frac{1}{2}|I_{xy}|^2\Re(Z_{xy}).$$

Power dissipation of the potential $(v(x), v(y)) \in \mathbb{C}$

$$\mathcal{P}[v]_{Z_{xy}} = \frac{1}{2}\frac{\Re(Z_{xy})}{|Z_{xy}|^2}|v(x) - v(y)|^2.$$

Power dissipation in graphs

Let $\mathcal{G} = (V, E)$ be a finite graph, $\mathcal{Z} = \{Z_{xy},\ \{x,y\} \in E\}$ a network on $\mathcal{G}$ and $\ell(V) = \{v\colon V \to \mathbb{C}\}$. The quadratic form

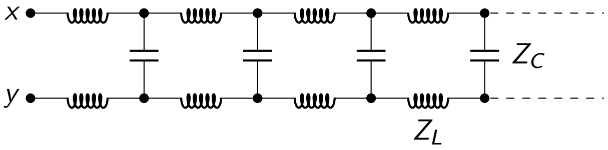
$$\mathcal{P}_{\mathcal{Z}}[v] = \frac{1}{2} \sum_{\{x,y\} \in E} \frac{\Re(Z_{xy})}{|Z_{xy}|^2} |v(x) - v(y)|^2$$

is the power dissipation in $\mathcal{G}$ associated with the network $\mathcal{Z}$.

- If $Z_{x,y}$, I_{xy}, v real, $\mathcal{P}_{\mathcal{Z}}(v) = \frac{1}{2} \sum_{\{x,y\} \in E} \frac{1}{Z_{xy}} (v(x) - v(y))^2$.

Power dissipation in an infinite network. The infinite ladder

Feynman's infinite ladder network [4]



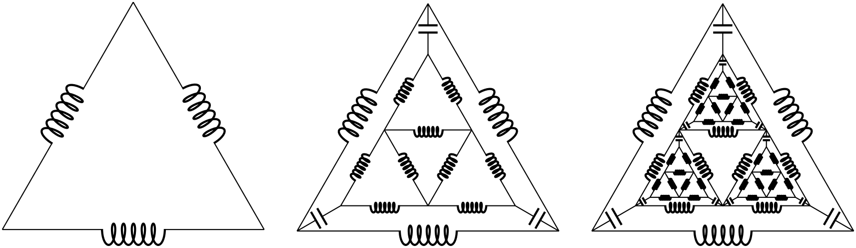
If $\omega^2 LC < 4$, the characteristic impedance of the circuit satisfies

$$\Re(Z_{xy}^{\text{eff}}) > 0$$

even though all elements in the circuit have purely imaginary impedances!

The Feynman-Sierpinski ladder

Infinite network $\mathscr{Z}_{\mathsf{FS}} = \{Z_{xy},\ \{x,y\} \in E_\infty\}$.



Capacitors $Z_C = \frac{1}{i\omega C}$, inductors $Z_L = i\omega L$.

Theorem [2]: The effective impedance of the Feynman-Sierpinski ladder has positive real part whenever

$$9(4-\sqrt{15}) < 2\omega^2 LC < 9(4+\sqrt{15}) \qquad \text{(FC)}$$

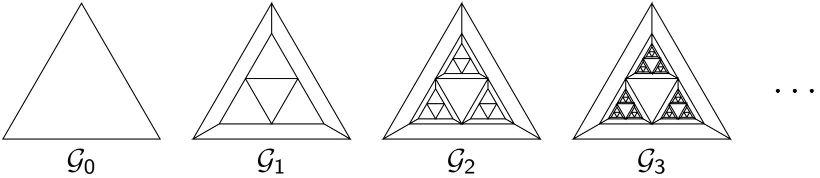
(filter condition).

In this case,

$$Z_{\mathsf{FS}}^{\mathsf{eff}} = \frac{1}{10\omega C}\left((9+2\omega^2 LC)i + \sqrt{144\omega^2 LC - 4(\omega^2 LC)^2 - 81}\right).$$

From infinite graphs to fractals

Underlying infinite graph structure $\mathcal{G}_\infty$ approximated by finite graphs $\mathcal{G}_n = (V_n, E_n)$, $n \geq 0$.



- $\pi\colon \mathcal{G}_\infty \to \mathbb{R}^2$
- $\pi(\mathcal{G}_0) \subseteq \pi_1(\mathcal{G}_1) \subseteq \ldots \subseteq \pi_n(\mathcal{G}_n) \subseteq \ldots$

The fractal Q_∞

The unique compact set $Q_\infty \subseteq \mathbb{R}^2$ such that

$$Q_\infty = \overline{\bigcup_{n \geq 0} \pi(\mathcal{G}_n)}^{\text{Eucl}}$$

is a fractal quantum graph.

The fractal K_∞

The set

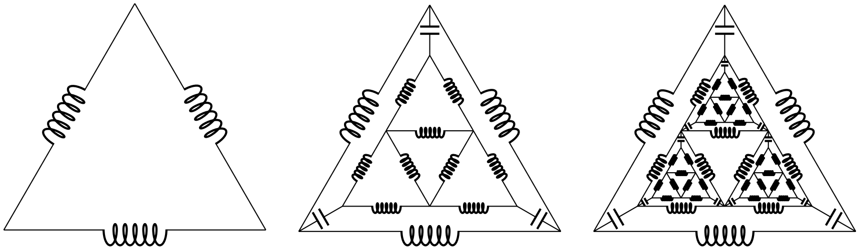
$$K_\infty = Q_\infty \setminus \bigcup_{n \geq 0} \pi(\mathring{E}_n)$$

is the union of countable many isolated points (nodes in V_*) and a Cantor dust C_∞ (accumulation points).

Observations/consequences

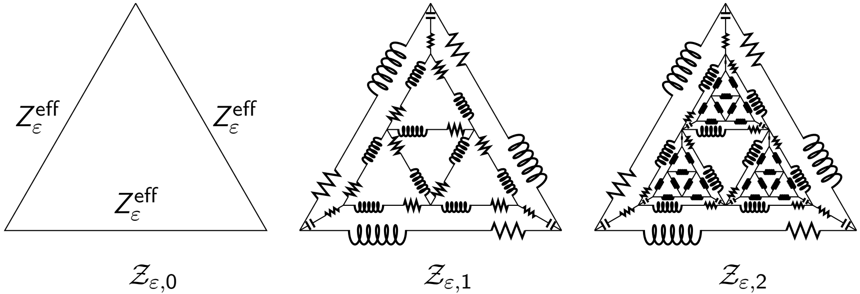
- Identify V_n with $\pi(V_n)$,
- $V_* = \bigcup_{n \geq 1} V_n$ is dense in K_∞,
- K_∞ is compact in the Euclidean topology.

Networks on $\mathcal{G}_n$



Networks on $\mathcal{G}_n$

$$\mathcal{Z}_{\varepsilon,n} = \{Z_{\varepsilon,xy} \mid \{x,y\} \in E_n\}, \qquad Z_{\varepsilon,xy} = Z_{xy} + \varepsilon.$$



(For completeness, $Z_\varepsilon^{\text{eff}} := \lim_{n\to\infty} Z_{\varepsilon,n}^{\text{eff}}$.)

Theorem [2]: Under (FC), the network $\mathcal{Z}_{\varepsilon,n}$ approximates the Sierpinski ladder $\mathcal{Z}$ in the sense that

$$\lim_{\varepsilon \to 0+} \lim_{n \to \infty} Z_{\varepsilon,n}^{\text{eff}} = Z_{\text{FS}}^{\text{eff}},$$

where $Z_{\varepsilon,n}^{\text{eff}}$ is the effective impedance of $\mathcal{Z}_{\varepsilon,n}$.

- Up to now, assume that (FC) holds.

Towards power dissipation in K_∞

The power dissipation in V_* associated with the Feynman-Sierpinski ladder is the quadratic form

$$\mathrm{P}_{\mathrm{FS}}[v] := \lim_{\varepsilon \to 0_+} \lim_{n \to \infty} \mathcal{P}_{\mathcal{Z}_{\varepsilon,n}}[v_{|V_n}],$$

where $\mathcal{P}_{\mathcal{Z}_{\varepsilon,n}} \colon \ell(V_n) \to \mathbb{R}$ is the power dissipation in $\mathcal{G}_n$ associated with $\mathcal{Z}_{\varepsilon,n}$.

$$\operatorname{dom} \mathsf{P}_{\mathsf{FS}} := \{v \in \ell(V_*) \mid \mathsf{P}_{\mathsf{FS}}[v] < \infty\}$$

- meaningful functions in this set?
- extension of functions?

Harmonic functions

- A function $h \in \ell(V_*)$ is harmonic if for any $\varepsilon > 0$

$$\mathsf{P}_{\mathcal{Z}_{\varepsilon,0}}[h_{|_{V_0}}] = \mathsf{P}_{\mathcal{Z}_{\varepsilon,n}}[h_{|_{V_n}}] \qquad \text{for all } n \geq 0.$$

- Notation: $\mathcal{H}_{\mathsf{FS}}(V_*) := \{h \in \ell(V_*) \text{ harmonic}\}$.

- For any $h \in \mathcal{H}_{\mathsf{FS}}(V_*)$

$$\mathsf{P}_{\mathsf{FS}}[h] = \lim_{\varepsilon \to 0_+} \mathsf{P}_{\mathcal{Z}_{\varepsilon,n}}[h_{|_{V_n}}].$$

Harmonic extension rule

Theorem [2]: For any $h \in \mathcal{H}_{\mathsf{FS}}(V_*)$, $j = 1, 2, 3$, $h_{|_{G_j(V_\mathbf{0})}} = A_j h_{|_{V_\mathbf{0}}}$, where

$$A_1 = \frac{1}{9Z_C + 5Z_{\mathsf{FS}}^{\mathsf{eff}}} \begin{pmatrix} 3Z_C + 5Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C & 3Z_C \\ 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + Z_{\mathsf{FS}}^{\mathsf{eff}} \\ 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} \end{pmatrix}$$

$$A_2 = \frac{1}{9Z_C + 5Z_{\mathsf{FS}}^{\mathsf{eff}}} \begin{pmatrix} 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + Z_{\mathsf{FS}}^{\mathsf{eff}} \\ 3Z_C & 3Z_C + 5Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C \\ 3Z_C + Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} \end{pmatrix}$$

$$A_3 = \frac{1}{9Z_C + 5Z_{\mathsf{FS}}^{\mathsf{eff}}} \begin{pmatrix} 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + Z_{\mathsf{FS}}^{\mathsf{eff}} & Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} \\ 3Z_C + Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} & 3Z_C + 2Z_{\mathsf{FS}}^{\mathsf{eff}} \\ 3Z_C & 3Z_C & 3Z_C + 5Z_{\mathsf{FS}}^{\mathsf{eff}} \end{pmatrix}.$$

Observations

- A_1, A_2, A_3 have the same eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = \frac{3Z_{\mathsf{FS}}^{\mathsf{eff}}}{9Z_C + 5Z_{\mathsf{FS}}^{\mathsf{eff}}}, \quad \lambda_3 = \frac{1}{3}\lambda_2,$$

- $\mathrm{span}\{u_1\} = \{\text{constant harmonic functions}\}$,
- $|\lambda_3| < |\lambda_2| < 1$. Otherwise, $\mathsf{P}_{\mathsf{FS}}[h] = \mathsf{P}_{\mathcal{Z}_0}[A_j h_{|_{V_0}}]$ (power dissipation concentrates in one single cell, a contradiction).

Continuity of harmonic functions

Theorem (A.R.'17): Harmonic functions are continuous on V_*.

Harmonic extension and power dissipation

Lemma: There exists $r \in (0,1)$ such that

$$\mathsf{P}_{\mathcal{Z}_0}[A_j h_0] \leq r^2 \, \mathsf{P}_{\mathcal{Z}_0}[h_0] \qquad \forall\, j = 1,2,3$$

and any non-constant function $h_0 \in \ell(V_0)$.

Consequences

- Harmonic functions are well-defined on K_∞,

$$\mathcal{H}_{\mathsf{FS}}(K_\infty) = \{h\colon K_\infty \to \mathbb{C} \mid h_{|V_*} \text{ harmonic on } V_*\}.$$

- Well-defined power dissipation in K_∞,

$$\mathsf{P}_{\mathsf{FS}}[h] = \mathsf{P}_{\mathsf{FS}}[h_{|V_*}], \qquad h \in \mathcal{H}_{\mathsf{FS}}(K_\infty).$$

Power dissipation measure

Theorem (A.R.'17): For each non-constant $h \in \mathcal{H}_{\mathsf{FS}}(K_\infty)$, power dissipation induces a continuous measure ν_h on K_∞ with $\operatorname{supp} \nu_h = C_\infty$.

Define

$$\nu_h(T_w) := \lim_{\varepsilon \to 0_+} \lim_{n \to \infty} \sum_{\substack{x,y \in T_w \cap V_n \\ \{x,y\} \in E_n}} \mathrm{P}_{\mathcal{Z}_{\varepsilon,n}}[h]_{xy}$$

for each m-cell T_w.

Oscillations

Corollary: For any m-cell T_w,

$$\nu_h(T_w) \asymp \operatorname{osc}(h_{|_{T_w}})^2.$$

Self-similar measure on K_∞

Bernouilli measure μ on K_∞:

$$\mu(T_{w_1 \dots w_n}) = \mu_{w_1} \cdots \mu_{w_n}, \qquad \sum_{i=1}^{3} \mu_i = 1.$$

- $\operatorname{supp} \mu = C_\infty$,
- (C_∞, μ) is probability space,
- take $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$.

Singularity of power dissipation

Theorem (A.R.'17): Assume that for any non-constant $h \in \mathcal{H}_{\mathsf{FS}}(K_\infty)$ such that $h_{|v_0} = v_0$

$$x \mapsto \|D_{\mathsf{P}_0} M_n(x) \dots M_1(x) v_0\|$$

is non-constant for some $n \geq 1$. Then, the measure ν_h is singular with respect to μ.

Summary

- Power dissipation on an infinite (fractal) AC network
- harmonic potentials are continuous
- (non-atomic) power dissipation measure
- singularity of power dissipation measure

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Thank you for your attention!