Notes on the class of Janowski Starlike Log-Harmonic Mappings of complex order "b"

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June 14, 2017

Abstract

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In this paper, we consider univalent log-harmonic mappings of the form

$$f(z) = zh(z)\overline{g(z)}$$

defined on the unit disk $\mathbb{D}$ which are starlike. Some distortion theorems are obtained.

1. Harmonic Functions

Definition

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathfrak{D}$ is said to be harmonic in $\mathfrak{D}$ if both u and v are real harmonic in $\mathfrak{D}$, that is, u, v satisfy, respectively the *Laplace equations*

$$\Delta u = u_{xx} + u_{yy} = 0, \quad \Delta v = v_{xx} + v_{yy} = 0$$

There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that

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There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that

$$u = \mathfrak{Re}(U) \text{ and } v = \mathfrak{Im}(V).$$

Therefore, it has a canonical decomposition

$$f = h + \overline{g} \tag{1}$$

where h and g are, respectively, the analytic functions

$$h = \frac{U+V}{2} \text{ and } g = \frac{U-V}{2}.$$

Example

$f(z) = z - 1/\bar{z} + 2\ln|z|$ is a harmonic univalent function form the exterior of the unit disc $\mathbb{D}$ onto $\mathbb{C}/\{0\}$, where $h(z) = z + \log z$ and $g(z) = \log z - 1/z$.

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It is well-known that if $f = u + iv$ has continuous partial derivatives, then f is analytic if and only if the Cauchy-Riemann equations are satisfied. It follows that every analytic function is a complex-valued harmonic function.

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It is well-known that if $f = u + iv$ has continuous partial derivatives, then f is analytic if and only if the Cauchy-Riemann equations are satisfied. It follows that every analytic function is a complex-valued harmonic function. However, not every complex-valued harmonic function is analytic.

- A subject of considerable importance in harmonic mappings is the Jacobian J_f of a function $f = u + iv$, defined by $J_f = u_x v_y - u_y v_x$. Or, in terms of f_z and $f_{\bar{z}}$, we have

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2,$$

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- When J_f is positive in $\mathcal{D}$, the harmonic function f is called orientation-preserving or sense-preserving in $\mathcal{D}$.
- An analytic univalent function is a special case of an sense-preserving harmonic univalent function. For analytic function f, it is well-know that $J_f \neq 0$ if and only if f is locally univalent at z.

For harmonic functions we have the following useful result due to Lewy

Theorem

A harmonic mapping is locally univalent in a neighborhood of a point z_0 if and only if the Jacobian $J_f(z) \neq 0$ at z_0.

Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc., 42(1936), 689-692.

- The first key insight into harmonic univalent mappings came from Clunie and S. Small, who observe that $f = h + \overline{g}$ is locally univalent and sense-preserving if and only if $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$ $(z \in \mathbb{D})$. This is equivalent to

$$|g'(z)| < |h'(z)| \quad (z \in \mathbb{D}). \tag{2}$$

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- The function $w = g'/h'$ is called the second dilatation of f. We denote the class of the second dilatation function of f by $\mathcal{W}$. Note that $|w(z)| < 1$.

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Class $\mathcal{S}_{\mathcal{H}}$

- We denote by $\mathcal{S}_{\mathcal{H}}$ the family of all harmonic, complex-valued, sense-preserving, normalized and univalent mappings defined on $\mathbb{D}$. Thus a function f in $\mathcal{S}_{\mathcal{H}}$ admits the representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{3}$$

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Definition of a Log-Harmonic Function

Let $\mathcal{H}(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = w f_z \left(\frac{\overline{f}}{f} \right), \tag{4}$$

where the second dilation function $w \in \mathcal{H}(\mathbb{D})$ is such that $|w(z)| < 1$ for all $z \in \mathbb{D}$.

[0] Z. Abdulhadi and D. Bshouty, Univalent functions in $H \cdot \overline{H}(\mathbb{D})$, *Tran. Amer. Math. Soc.*, **305**(2) (1988), 841-849.

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- Observe that nonconstant log-harmonic functions are sense-preserving on $\mathbb{D}$.

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Harmonic and Log-Harmonic Functions

The motivation behind the study of log-harmonic functions comes from the fact that for any sense preserving harmonic function $u = H_1 + \overline{G}_1$, H_1 and $\overline{G}_1$ in $\mathcal{H}(\mathbb{D})$, e^u is a non-vanishing function of $\mathcal{H} \cdot \overline{\mathcal{H}}(\mathbb{D})$. Thus, of particular interest are those functions of $H \cdot \overline{H}(\mathbb{D})$ that vanish in $\mathbb{D}$, as their zeros correspond to some singularities of harmonic functions.

It has been shown that if f is non-vanishing log-harmonic mapping in $\mathbb{D}$, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{5}$$

where h and g are analytic in $\mathbb{D}$.

On the other hand if f vanishes at $z = 0$, but not identically zero then f admits the following representation

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \tag{6}$$

where $\mathrm{Re}\beta > -1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with the normalization $h(0) \neq 0$, $g(0) = 1$.

Example

$f(z) = z|z|^{2\beta}$, Re$\beta > -\frac{1}{2}$ and $f(1) = 1$ is a solution of the equation

$$\overline{f_{\bar{z}}} = wf_z\left(\frac{\bar{f}}{f}\right)$$

in $\mathbb{C}$ with $w \equiv \bar{\beta}/(1+\beta)$

We also note that univalent log-harmonic mappings have been studied extensively in

1. Z. Abdulhadi, Close-to-starlike log-harmonic mappings, *Internat. J. Math. and Math. Sci.*, **19**(3) (1996), 563-574.
2. Z. Abdulhadi, Typically real logharmonic mappings, *Internat. J. Math. and Math. Sci.*, **31**(1) (2002), 1-9.
3. Z. Abdulhadi and Y. Abu Muhanna, Starlike Log-harmonic Mappings of Order α, *J. Inequal. Pure and Appl. Math.*, **7**(4) (2006), Article 123.
4. Z. Abdulhadi and W. Hengartner, Spirallike logharmonic mappings, *Complex Variables Theory Appl.*, **9**(2-3) (1987), 121-130.
5. Z. Abdulhadi and W. Hengartner, One pointed univalent logharmonic mappings, *J. Math. Anal. Appl.*, **203**(2) (1996), 333-351.

Starlike Log-Harmonic Functions

Let $f(z) = zh(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping if

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \text{Re}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right) > 0 \tag{7}$$

for every $z \in \mathbb{D}$. The class of all starlike log-harmonic functions is denoted by $\mathcal{S}^*_{LH}$.

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for every $z \in \mathbb{D}$. The class of all starlike log-harmonic functions is denoted by $\mathcal{S}^*_{LH}$.

- If $f \in \mathcal{S}^*_{LH}$ then $F(\zeta) = \log(f(e^{\zeta}))$ is univalent and harmonic on the half plane $\{\zeta : \mathrm{Re}\zeta < 0\}$.

[0] Z. Abdulhadi and Y. Abu Muhanna, Starlike Log-harmonic Mappings of Order α, *J. Inequal. Pure and Appl. Math.*, **7**(4) (2006), Article 123.

Jacobian of a Log-Harmonic Functions

The Jacobian of a log-harmonic function of the form

$$f(z) = zh(z)\overline{g(z)}$$

is defined by

$$J_f(z) = |f(z)|^2 \left(\left| \frac{1}{z} + \frac{h'(z)}{h(z)} \right|^2 - \left| \frac{g'(z)}{g(z)} \right|^2 \right) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.$$

for all z in $\mathbb{D}$.

Subordination Principle

Let Ω be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and let $s_1(z) = z + a_2 z^2 + \cdots$, $s_2(z) = z + b_2 z^2 + \cdots$ be analytic functions in $\mathbb{D}$. We say that $s_1(z)$ is subordinate to $s_2(z)$ if there exist $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ and it is denoted by $s_1(z) \prec s_2(z)$.

Starlike Univalent Functions

Let $\varphi(z)$ be analytic function in $\mathbb{D}$ with the normalization $\varphi(0) = 0$, $\varphi'(0) = 1$. If $\varphi(z)$ satisfies the condition

$$\text{Re}\left(z\frac{\varphi'(z)}{\varphi(z)}\right) > 0 \tag{8}$$

for every $z \in \mathbb{D}$, then $\varphi(z)$ is called starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$.

Some Theorems

We note that in our proofs we will need the following theorems:

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Theorem (1.1)

Let $\varphi(z)$ be an element of $\mathcal{S}^*$, then

$$\frac{1-r}{1+r} \leq \left| z\frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{1+r}{1-r} \quad (|z| = r < 1). \tag{9}$$

A.W. Goodman, *Univalent Functions*, Vol 1, Mariner Publishing Comp. Inc., Washington, New Jersey, 1983.

Some Theorems

Theorem (1.2)

$f(z) = zh(z)\overline{g(z)}$ be a log-harmonic function on $\mathbb{D}$, $0 \notin hg(\mathbb{D})$. Then $f \in \mathcal{S}^*_{LH}$ if and only if $\varphi(z) = \left(z\frac{h(z)}{g(z)}\right) \in \mathcal{S}^*$.

Z. Abdulhadi and Y. Abu Muhanna, Starlike Log-harmonic Mappings of Order α, *J. Inequal. Pure and Appl. Math.*, **7**(4) (2006), Article 123.

Some Theorems

Theorem (1.3)

Let $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}$*, with* $w(0) = 0$*. Then we have*

$$re^{-\frac{4r}{1+r}} \leq |f(z)| \leq re^{\frac{4r}{1-r}} \tag{10}$$

for all $|z| = r < 1$*. The equalities occur if and only if* $f(z) = \bar{\zeta} f_0(\zeta z)$, $|\zeta| = 1$, *where*

$$f_0(z) = z\left(\frac{1-\bar{z}}{1-z}\right)e^{Re\frac{4z}{1-z}}.$$

Z. Abdulhadi and Y. Abu Muhanna, Starlike Log-harmonic Mappings of Order α, *J. Inequal. Pure and Appl. Math.*, **7**(4) (2006), Article 123.

2. Some Results on the class of Janowski Starlike Log-Harmonic Mappings of complex order "b"

Definition

Let $f = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{LH}$. We say that f is a Janowski starlike log-harmonic mapping if

$$1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right) = p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, p(z) \in \mathcal{P}(A, B) \tag{11}$$

where $-1 \leq B < A \leq 1$, $b \neq 0$ and complex and denote by $\mathcal{S}^*_{LH}(A, B, b)$ the set of all starlike log-harmonic mappings. Also we denote $\mathcal{S}^*_{PLH}(A, B, b)$ the class of all functions in $\mathcal{S}^*_{LH}(A, B, b)$ for which $(zh(z)) \in \mathcal{S}^*$ for all $z \in \mathbb{D}$.

Introduction

We note that if we give special values to b, then we obtain important subclasses of Janowski starlike log-harmonic mappings

i. For $b = 0$, we obtain the class of starlike log-harmonic mappings.

ii. For $b = 1 - \alpha$, $0 \leq \alpha < 1$, we obtain the class of starlike log-harmonic mappings of order α.

iii. For $b = e^{-i\lambda} cos\lambda$, $|\lambda| < \frac{\pi}{2}$, we obtain the class of $\lambda-$ spirallike log-harmonic mappings.

iv. For $b = (1 - \alpha)e^{-i\lambda} cos\lambda$, $0 \leq \alpha < 1$, $|\lambda| < \frac{\pi}{2}$, we obtain the class of $\lambda-$ spirallike log-harmonic mappings of order α.

Theorem

(2.1) *Let* $f = zh(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^*_{PLH}(A, B, b)$. *Then*

$$f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A, B, b) \Leftrightarrow \begin{cases} z\dfrac{h'(z)}{h(z)} - \overline{z}\dfrac{\overline{g'(z)}}{\overline{g(z)}} \prec \dfrac{b(A-B)z}{1+Bz}; & B \neq 0, \\ z\dfrac{h'(z)}{h(z)} - \overline{z}\dfrac{\overline{g'(z)}}{\overline{g(z)}} \prec bAz; & B = 0. \end{cases} \quad (12)$$

Proof.

Let $f \in \mathcal{S}^*_{LH}(A, B, b)$. Using the principle of subordination then we have

$$1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right) = 1 + \frac{1}{b}\left(z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}\right)$$

$$= \begin{cases} \dfrac{1 + A\phi(z)}{1 + B\phi(z)}; & B \neq 0, \\ 1 + A\phi(z); & B = 0, \end{cases}$$

$$\Leftrightarrow z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}} = \begin{cases} \frac{b(A-B)\phi(z)}{1+B\phi(z)}; & B \neq 0, \\ bA\phi(z); & B = 0, \end{cases}$$

$$\Leftrightarrow z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}} \prec \begin{cases} \frac{b(A-B)z}{1+Bz}; & B \neq 0, \\ bAz; & B = 0. \end{cases}$$

Theorem

(2.2) *Let* $f = zh(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^*_{PLH}(A, B, b)$*. Then*

$$\begin{cases} G(A,B,-r) \leq \left|\frac{h(z)}{g(z)}\right| \leq G(A,B,r); & B \neq 0, \\ G_1(A,-r) \leq \left|\frac{h(z)}{g(z)}\right| \leq G_1(A,r); & B = 0, \end{cases}$$

where

$$\begin{cases} G(A,B,r) = \dfrac{(1+Br)^{\frac{(A-B)(|b|-Reb)}{2B}}}{(1-Br)^{\frac{(A-B)(|b|+Reb)}{2B}}}; & B \neq 0, \\ G_1(A,r) = \dfrac{(1+r)^{\frac{A|b|}{2}}}{(1-r)^{\frac{A|b|}{2}}}; & B = 0. \end{cases} \tag{13}$$

Proof.

The function $\left(\frac{1+Az}{1+Bz}\right)$ maps $|z| = r$ on to crcle with the centre $C(r) = (\frac{1-ABr^2}{1-B^2r^2}, 0)$ and the raidus $p(r) = \frac{(A-B)r}{1-B^2r^2}$. Therefore using the definition of subordination and Theorem (2.1), we get

$$\begin{cases} \left|\left(1 + \frac{1}{b}\left(z\frac{h'(z)}{h(z)} - \overline{z\frac{g'(z)}{g(z)}}\right) - \frac{1-ABr^2}{1-B^2r^2}\right)\right| \leq \frac{(A-B)r}{1-B^2r^2}; & B \neq 0, \\ \left|\left(1 + \frac{1}{b}\left(z\frac{h'(z)}{h(z)} - \overline{z}\frac{g'(z)}{g(z)}\right) - 0\right)\right| \leq Ar; & B = 0. \end{cases} \tag{14}$$

The inequality (14) takes in the form,

$$\begin{cases} \frac{(B(A-B)Reb)r^2-|b|(A-B)r}{1-B^2r^2} \leq Re\left(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) \\ \leq \frac{b(A-B)r}{1-B^2r^2} + \frac{(B(A-B)Reb)r^2}{1-B^2r^2}; & B \neq 0, \\ -\frac{A|b|r}{1-r^2} \leq Re(z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}) \leq \frac{A|b|r}{1-r^2}; & B = 0. \end{cases} \tag{15}$$

On the other hand we have,

$$Re\left(z\frac{h'(z)}{h(z)}-z\frac{g'(z)}{g(z)}\right)=r\frac{\partial}{\partial r}\left(\log|h(z)|-\log|g(z)|\right).$$

Thus the inequality (15) can be written in the form,

$$\begin{cases} \frac{[B(A-B)Reb.r-|b|(A-B)]}{(1-Br)(1+Br)}\leq\frac{\partial}{\partial r}\log|h(z)-g(z)| \\ \leq\frac{[B(A-B)Reb.r+|b|(A-B)]}{(1-Br)(1+Br)}; & B\neq 0, \\ -\frac{A|b|}{(1-r)(1+r)}\leq\frac{\partial}{\partial r}\log|h(z)-g(z)|\leq\frac{A|b|}{(1-r)(1+r)}; B=0, \end{cases} \tag{16}$$

integrating both sides of (16) from 0 to r we get (13).

Corollary

(2.3) *The radius of starlikeness of the class* $\mathcal{S}^*_{PLH}$ *is*

$$r_s = \begin{cases} \frac{2}{(A-B)|b|+\sqrt{(A-B)^2|b|^2+4[B^2+(AB-B^2)Reb]}}; & B \neq 0, \\ \frac{1}{|b|A}; & B = 0. \end{cases} \quad (17)$$

Proof.

The inequality (14) can be written in the form

$$\begin{cases} \left| \frac{zf_{;z} - \overline{z} f_{\overline{z}}}{f} - \left[\frac{1 - \left(B^2 + (AB - B^2 Reb) r^2 - i((AB - B^2) Imb) r^2 \right)}{1 - B^2 r^2} \right] \right| \leq \frac{|b|(A-B)r}{1 - B^2 r^2}; & B \neq 0 \\ \left| \frac{zf_z - \overline{z} f_{\overline{z}}}{f} - 1 \right| \leq |b| \, Ar; & B = 0. \end{cases}$$

Therefore we have

$$Re \left(\frac{zf_z - \overline{z} f_{\overline{z}}}{f} \right) \geq \begin{cases} \frac{1 - (A-B)|b| r - (B^2 + (AB - B^2) Reb) r^2}{1 - B^2 r^2}; & B \neq 0, \\ 1 - |b| \, Ar; & B = 0, \end{cases}$$

which gives (17).

Lemma

(2.4) *Let* $f = z\,|z|^{2\beta}\,h(z)\overline{g(z)} \in \mathcal{S}_{LH}$ *and let* $w(z)$ *be the second dilatation of* f. *Then*

$$\frac{||\beta| - |\beta+1|\,r|}{||\beta+1| - |\beta|\,r|} \leq |w(z)| \leq \frac{||\beta| + |\beta+1|\,r|}{||\beta+1| + |\beta|\,r|}. \qquad (18)$$

This inequality is sharp because the extremal function is

$$w(z) = e^{i\theta}\frac{e^{i\ell}z - \left|\frac{\overline{\beta}}{\beta+1}\right|}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\ell}z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}.$$

Proof.

$f = z|z|^{2\beta} h(z)\overline{g(z)} \in \mathcal{S}_{LH}$ and let $1^{2\beta} = 1$. Then f is the solution of the nonlinear elliptic partial differantial equation

$$w(z) = \frac{\overline{f_{\overline{z}}}}{\overline{f}} \cdot \frac{f}{f_z}$$

$$f_z = \left(\frac{1}{z} + \frac{\beta}{z} + \frac{h'(z)}{h(z)} \right) f,$$

$$f_{\overline{z}} = \left(\frac{\overline{\beta}}{z} + \frac{g'(z)}{g(z)} \right) \overline{f}$$

$$w(z) = \frac{\overline{f_{\overline{z}}}}{\overline{f}} \cdot \frac{f}{f_z} = \frac{\overline{\beta} + z\frac{g'(z)}{g(z)}}{(\beta + 1) + z\frac{h'(z)}{h(z)}}, \quad w(0) = \frac{\overline{\beta}}{\beta + 1}, \quad |w(0)| < 1.$$

On the other hand for $Re\beta > -\frac{1}{2}$, we have $\left|\frac{\overline{\beta}}{\beta+1}\right| < 1$. Therefore we can take $w(0) = c_0 = \left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\theta}, \quad \theta \in \mathbb{R}$.
Now consider the function

$$\phi(z) = \frac{e^{-i\theta}w(z) - \left|\frac{\overline{\beta}}{\beta+1}\right|}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\theta}w(z)}, \quad z \in \mathbb{D},$$

which satisfies the conditions Schwarz lemma and use the estimate $|\phi(z)| \leq |z| < r$, to get

$$\left|e^{-i\theta}w(z) - \left|\frac{\overline{\beta}}{\beta+1}\right|\right| \leq r\left|\left|\frac{\overline{\beta}}{\beta+1}\right| e^{-i\theta}w(z) - 1\right|.$$

This is equivalent to

$$\left|w(z) - \frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2}\right| \leq \frac{r\left(1 - \left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} \qquad (19)$$

and the equality holds only for a function of the form

$$w(z) = e^{i\theta}\frac{e^{i\ell}z - \left|\frac{\overline{\beta}}{\beta+1}\right|}{1 - \left|\frac{\overline{\beta}}{\beta+1}\right| e^{i\ell}z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}.$$

From the inequality (19) we have then

$$|w(z)| = \left|e^{-i\theta}w(z)\right| \geq \left|\frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} - \frac{r\left(1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2}\right| = \frac{\left|\left|\frac{\overline{\beta}}{\beta+1}\right| - r\right|}{1-\left|\frac{\overline{\beta}}{\beta+1}\right| r}$$

$$|w(z)| = \left|e^{-i\theta}w(z)\right| \leq \frac{\left|\frac{\overline{\beta}}{\beta+1}\right|(1-r^2)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} - \frac{r\left(1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2\right)}{1-\left|\frac{\overline{\beta}}{\beta+1}\right|^2 r^2} = \frac{\left|\frac{\overline{\beta}}{\beta+1}\right| + r}{1+\left|\frac{\overline{\beta}}{\beta+1}\right| r}.$$

Theorem

(2.5) *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{PLH}(A, B, b)$. *Then*

$$-\frac{1+r}{1-r} \leq \left|\frac{g'(z)}{g(z)}\right| \leq \frac{1+r}{1-r}. \tag{20}$$

Proof.

Since $\varphi = (zh(z)) \in \mathcal{S}^*$. Then

$$\frac{1-r}{1+r} \leq \left| z\frac{\varphi'(z)}{\varphi(z)} \right| = \left| 1 + z\frac{h'(z)}{h(z)} \right| \leq \frac{1+r}{1-r}. \tag{21}$$

On the other hand, if we take $\beta = 0$ in Lemma (2.4), then we have

$$-r \leq |w(z)| = \left| \frac{z\frac{g'(z)}{g(z)}}{1 + z\frac{h'(z)}{h(z)}} \right| \leq r$$

,

$$-r\left| 1 + z\frac{h'(z)}{h(z)} \right| \leq \left| z\frac{g'(z)}{g(z)} \right| \leq r\left| 1 + z\frac{h'(z)}{h(z)} \right| \tag{22}$$

using (21) in the inequality (22) we obtain (20).

Theorem

(2.6) *Let* $f = zh(z)\overline{g(z)}$ *be an element of* $\mathcal{S}^{*}_{PLH}(A, B, b)$ *then*

$$F(A, B, -r) \leq |g(z)| \leq F(A, B, r),$$

$$F_1(A, -r) \leq |g(z)| \leq F_1(A, r),$$

where

$$F(A, B, r) = \frac{1}{(1-r)^2} \cdot \frac{(1+Br)^{\frac{(A-B)(|b|+Reb)}{2B}}}{(1-Br)^{\frac{(A-B)(|b|-Reb)}{2B}}}$$

$$F_1(A, r) = \frac{(1+r)^{\frac{|b|A}{2}}}{(1-r)^{\frac{|b|A}{2}+2}}.$$

Proof.

$f \in \mathcal{S}^*_{LH}(A,B,b)$, $h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, $h(0) = 1 \neq 0$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, $g(0) = 1 \neq 0$. Therefore if $(zh(z))$ is starlike then we have

$$Re\left(z\frac{(zh(z))'}{zh(z)}\right) > 0 \Rightarrow z\frac{(zh(z))'}{zh(z)} = p(z)$$

where $p(z) \in \mathcal{P}$, which yields

$$\left|z\frac{(zh(z))'}{zh(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2} \tag{23}$$

After simple calculations from (23) we get

$$\frac{1-r}{1+r} \leq Re\left(z\frac{(zh(z))'}{zh(z)}\right) \leq \frac{1+r}{1-r} \tag{24}$$

On the other hand we have

$$Re\left(z\frac{(zh(z))'}{zh(z)}\right) = r\frac{\partial}{\partial r}\log|zh(z)|\,.$$

Therefore the inequality (24) can be written in the form

$$\frac{1-r}{r(1+r)} \leq \frac{\partial}{\partial r} \log |zh(z)| \leq \frac{1+r}{r(1-r)} \tag{25}$$

and upon integration of both sides (25) from 0 to r, we get

$$\frac{1}{(1+r)^2} \leq |h(z)| \leq \frac{1}{(1-r)^2} \tag{26}$$

using Theorem (2.5) and the inequality (25) we get the result.

Corollary

(2.7) *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A,B,b)$ *and let* $(zh(z)) \in \mathcal{S}^*$*. Then*

$$\frac{1-r}{1+r} \le |h(z) + zh'(z)| \le \frac{1+r}{(1-r)^3} \tag{27}$$

Proof.

Follows immediately from Theorem (2.5). □

Corollary

(2.8) *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A, B, b)$ *and let* $(zh(z)) \in \mathcal{S}^*$. *Then*

$$\frac{1-r}{(1+r)^3}\left(\frac{1-r}{1+r}\right)^{\frac{|b|A}{2}} \leq |g'(z)| \leq \left(\frac{1-r}{1+r}\right)^{\frac{-|b|A}{2}} \frac{1+r}{(1-r)^3}. \qquad (28)$$

Proof.

Follows immediately from Theorem (2.5). □

Corollary

(2.9) *Let* $f = zh(z)\overline{g(z)} \in \mathcal{S}^{*}_{PLH}(A,B,b)$. *Then*

$$\begin{cases} \frac{r}{(1+r)^2}.F(A,B,-r) \le |f| \le \frac{r}{(1-r)^2}F(A,B,r); & B \neq 0, \\ \frac{r}{(1+r)^2}.F_1(A,-r) \le |f| \le \frac{r}{(1-r)^2}F_1(A,r); & B = 0. \end{cases} \tag{29}$$

Proof.

This result is a simple consequence of Theorem (2.5). □

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3. Some Coefficient Inequalities Of Janowski Starlike Log-harmonic Mappings Of Complex Order b

Theorem (3.1)

$F = z.\left|z\right|^{2\beta}.H(z).\overline{G(z)} \in S_{LH}$ *and If;*

$$1+\frac{1}{b}\left(\frac{zF_z - \bar{z}F_{\bar{z}}}{F}-1\right) = 1+\frac{1}{b}\left(z\frac{H'(z)}{H(z)}-\bar{z}.\frac{\overline{G'(z)}}{\overline{G(z)}}\right) = \begin{cases} \frac{1+A\phi(z)}{1+B\phi(z)}, & B \neq 0; \\ 1+A\phi(z), & B = 0; \end{cases}$$

Proof.

Let $F = z.\left|z\right|^{2\beta}.H(z).\overline{G(z)} \in S_{LH}$ and take a logarithm of both sides;

$$\log F = \log z.\left|z\right|^{2\beta}.H(z).\overline{G(z)} \in S_{LH}$$

$$\log F = \log z + \beta \log z + \beta \log \overline{z} + \log H(z) + \log \overline{G(z)} ......(2.1)$$

at *(2.1)* taking logarithmic derivatives first take to z after take to $\overline{z}$

$$F_z = F\left(\frac{1}{z} + \frac{\beta}{z} + \frac{H'(z)}{H(z)}\right)......(2.2)$$

$$F_{\overline{z}} = F\left(\frac{\beta}{\overline{z}} + \frac{\overline{G'(z)}}{\overline{G(z)}}\right)......(2.3)$$

Our class $f = z.h(z).\overline{g(z)}$ is a log-harmonic mapping;

$$1 + \frac{1}{b}\left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f} - 1\right) = \begin{cases} \frac{1+A\phi(z)}{1+B\phi(z)}, & B \neq 0; \\ 1 + A\phi(z), & B = 0; \end{cases}$$

Therefore; if $\beta \neq 0$; $F = z.\left|z\right|^{2\beta}.H(z).\overline{G(z)}$ is the most general form of log-harmonic mappings and at *(2.2)* ; *(2.3)* if we make simple calculations; we get the result.

Lemma (3.2)

Let $f = z|z|^{2\beta} h(z)\overline{g(z)} \in \mathcal{S}_{LH}$*.* $Re\beta > -\frac{1}{2}$; $h(z)$ *and* $g(z)$ *are both analytic in* $\mathbb{D}$*. Also* $g(0) = 1$ *and* $h(0) \neq 0$ *conditions are satisfied. Therefore*

$$Re\frac{h(z)}{g(z)} > 0 \Leftrightarrow Re\frac{f(z)}{z|z|^{2\beta}} > 0 \ldots (2.4)$$

satisfied.

Proof.

Let $f = z\left|z\right|^{2\beta} h(z)\overline{g(z)} \in \mathcal{S}_{LH}$

$$Re\frac{f(z)}{z\left|z\right|^{2\beta}} > 0 \Rightarrow 0 < Re\frac{\left|z\right|^{2\beta} h(z)\overline{g(z)}}{z\left|z\right|^{2\beta}} = Reh(z)\overline{g(z)}$$

$$= Re\frac{h(z)\overline{g(z)}g(z)}{g(z)} = Re\frac{h(z)\left|g(z)\right|^{2}}{g(z)} = \left|g(z)\right|^{2}.Re\frac{h(z)}{g(z)}$$

satisfied.

$$0 < \left|g(z)\right|^{2}.Re\frac{h(z)}{g(z)} \Rightarrow Re\frac{h(z)}{g(z)} > 0 ...(2.5)$$

satisfied. On the contrary;

Proof.

$$Re\frac{h(z)}{g(z)} > 0 \Rightarrow Re\frac{h(z)\left|g(z)\right|^2}{g(z)} > 0 \Leftrightarrow Re\frac{h(z)\overline{g(z)}g(z)}{g(z)} > 0$$

$$Reh(z).\overline{g(z)} > 0 \Rightarrow Re\frac{\left|z\right|^{2\beta} h(z)\overline{g(z)}}{z\left|z\right|^{2\beta}} > 0 ... (2.6)$$

satisfied. If we use *(2.5)* and *(2.6)* ;we take the expression of *(2.4)*.

Lemma (3.3)

$f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A, B, b)$ and $\frac{h(z)}{g(z)} = p(z)$. Here $h(z)$, $g(z)$, $p(z)$ functions are all analytic at $\mathbb{D}$. And their Taylor formulas are ;
$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ and
$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$;

$$|a_n| \leq 2\sum_{k=0}^{n-1} |b_k| + |b_n| \,; |b_0| = 1$$

Proof.

Take $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A, B, b)$. Then
$h(z) = 1 + a_1 z + a_2 z^2 + ... + a_n z^n$; $g(z) = 1 + b_1 z + b_2 z^2 + ... + b_n z^n$;
$p(z) = 1 + p_1 z + p_2 z^2 + ... + p_n z^n$ are like this. Here $\frac{h(z)}{g(z)} = p(z) \Rightarrow$

$$\Rightarrow (1+a_1 z+a_2 z^2+...+a_n z^n) = (1+p_1 z+p_2 z^2+...+p_n z^n).(1+b_1 z+b_2 z^2+...+b_n z^n$$

satisfied. If we make necessary calculations,
$1 + a_1 z + a_2 z^2 + ... + a_n z^n = 1 + (b_1 + p_1)z + (b_2 + p_1 b_1 + p_2)z^2 + (b_3 + p_1 b_2 + p_2 b_1 + p_3)z^3 + (b_4 + p_1 b_3 + p_2 b_2 + p_3 b_1 + p_4)z^4 + (b_5 + p_1 b_4 + p_2 b_3 + p_3 b_2 + p_4 b_1 + p_5)z^5 + ... + (b_n + p_1 b_{n-1} + p_2 b_{n-2} + ... + p_n)z^n + ...(2.7)$

Proof.

we get the expression . In this expression; If we make an equality between both sides of the coefficients; and then take their absolute values;

$|a_1| = |b_1 + p_1|$

$|a_2| = |b_2 + p_1 b_1 + p_2|$

$|a_3| = |b_3 + p_1 b_2 + p_2 b_1 + p_3|$

$|a_4| = |b_4 + p_1 b_3 + p_2 b_2 + p_3 b_1 + p_4|$

$|a_5| = |b_5 + p_1 b_4 + p_2 b_3 + p_3 b_2 + p_4 b_1 + p_5|$

....

$|a_n| = |b_n + p_1 b_{n-1} + p_2 b_{n-2} + p_3 b_{n-3} + ... + p_n|$

Here If we use $p_n \leq 2$ at all of the equalities

Proof.

$$|a_1| \leq 2 + |b_1|$$
$$|a_2| \leq 2 + 2|b_1| + |b_2|$$
$$|a_3| \leq 2 + 2|b_1| + 2|b_2| + |b_3|$$
$$|a_4| \leq 2 + 2|b_1| + 2|b_2| + 2|b_3| + |b_4|$$
$$|a_5| \leq 2 + 2|b_1| + 2|b_2| + 2|b_3| + 2|b_4| + |b_5|$$

....

$$|a_n| \leq 2 + 2|b_1| + 2|b_2| + 2|b_3| + 2|b_4| + 2|b_5| + \ldots + |b_n|$$

From here take to paranthesis of 2 and then we take the result.

Theorem (3.4)

$f = z|z|^{2\beta} h(z)\overline{g(z)} \in \mathcal{S}_{LH}$ and $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A, B, b)$ and $\frac{f}{z.|z|^{2\beta}} = p(z)$ is. Take $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then

$$1 + \frac{1}{b}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1\right) = 1 + \frac{1}{b} z \frac{p'(z)}{p(z)}$$

we get the result.

Proof.

Let $f = z\,|z|^{2\beta}\, h(z)\overline{g(z)} \in \mathcal{S}_{LH}$; Using *Lemma (2.3)*
$Re\frac{h(z)}{g(z)} > 0 \Leftrightarrow Re\frac{f}{z.|z|^{2\beta}}$ satisfied. Then; take $\frac{f(z)}{z.|z|^{2\beta}} = p(z)$ and from this expression
$f = z\,|z|^{2\beta}\,.p(z)$ get the result. First take logarithm of both sides ;

$$\log f = \log z + \beta \log z + \beta \log \overline{z} + \log p ...(2.8)$$

At *(2.5)* take derivatives first to z them multiply by z ;

$$\frac{f_z}{f} = \frac{1}{z} + \frac{\beta}{z} + \frac{p'}{p}$$

$$z\frac{f_z}{f} = 1 + \beta + z\frac{p'}{p} ...(2.9)$$

Proof.

Now at *(2.8)* take derivatives to $\overline{z}$ and multiply both sides by $\overline{z}$

$$\frac{f_{\overline{z}}}{f} = \beta \frac{1}{\overline{z}}$$

$$\overline{z}\frac{f_{\overline{z}}}{f} = \beta ...(2.10)$$

If we substract from *(2.6)* to *(2.10)*

$$\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = 1 + z\frac{p'}{p} ...(2.11)$$

we take this.
At expression of *(2.11)* multiply both sides by $\frac{1}{b}$ and then add 1 ; then we take the result.

Theorem (3.5)

$f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A, B, b)$. $s(z) = 1 + \frac{1}{b}(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1)$ *and*
$s(z) = 1 + \sum_{n=1}^{\infty} s_n z^n$;
$|s_1| \leq \frac{2}{|b|}, |s_2| \leq \frac{8}{|b|}, |s_3| \leq \frac{26}{|b|}, |s_4| \leq \frac{80}{|b|}, |s_5| \leq \frac{202}{|b|}$

Proof.

Let $f = zh(z)\overline{g(z)} \in \mathcal{S}^*_{LH}(A, B, b)$ and from *Teorem(2.5)*;

$s(z) = 1 + \frac{1}{b}(\frac{zf_z - \overline{z}f_{\overline{z}}}{f} - 1) = 1 + \frac{1}{b}z\frac{p'(z)}{p(z)}$

$\Rightarrow b.p(z) + z.p'(z) = b.p(z).s(z).....(2.12)$ satisfied.

$p(z) = 1 + p_1z + p_2z^2 + ... + p_nz^n...(2.13)$

$s(z) = 1 + s_1z + s_2z^2 + ... + s_nz^n...(2.14)$

(2.13) and *(2.14)* if we multiply them by b ;

$b.p(z).s(z) = b + b(s_1 + p_1)z + b(s_2 + p_1s_1 + p_2)z^2 + b(s_3 + p_1s_2 + p_2s_1 + p_3)z^3 + b(s_4 + p_1s_3 + p_2s_2 + p_3s_1 + p_4)z^4 + ... + b(s_{n-1} + p_1s_{n-2} + p_2s_{n-3} + p_3s_{n-4} + p_4s_{n-5} + ... + p_{n-1})z^{n-1} + b(s_n + p_1s_{n-1} + p_2s_{n-2} + p_3s_{n-3} + p_4s_{n-4} + p_{n-1}s_1 + p_n)z^n + b(s_{n+1} + p_1s_n + p_2s_{n-1} + p_3s_{n-2} + p_4s_{n-3} + p_{n-1}s_2 + p_ns_1 + p_{n+1})z^{n+1} + ...(2.15)$

Proof.

On the other hand;

$b.p(z) + z.p'(z) = b(1 + p_1 z + p_2 z^2 + p_3 z^3 + ... + p_{n-1} z^{n-1} + p_n z^n + p_{n+1} z^{n+1} + ...) + z(p_1 + 2p_2 z + 3p_3 z^2 + 4p_4 z^3 + ... + (n-1)p_{n-1} z^{n-2} + np_n z^{n-1} + (n+1)p_{n+1} z^n + (n+2)p_{n+2} z^{n+1} + ...(2.16)$

$=$

$b + bp_1 z + bp_2 z^2 + bp_3 z^3 + ... + bp_{n-1} z^{n-1} + bp_n z^n + bp_{n+1} z^{n+1} + ... + p_1 z + 2p_2 z^2 + 3p_3 z^3 + ... + (n-1)p_{n-1} z^{n-1} + np_n z^n + (n+1)p_{n+1} z^{n+1} + ...(2.17)$

Proof.

(2.17) can be written;

$b.p(z)+z.p'(z) = b+(p_1+bp_1)z+(2p_2+bp_2)z^2+(3p_3+bp_3)z^3+....+((n-1)p_{n-1}+bp_{n-1})z^{n-1}+(np_n+bp_n)z^n+((n+1)p_{n+1}+bp_{n+1})z^{n+1}+.....(2.18)$

If we make an equality between (2.15) and (2.18) then;

$b(s_1 + p_{1)} = p_1 + bp_1$

$b(s_2 + s_1 p_1 + p_2) = 2p_2 + bp_2$

$b(s_3 + s_2 p_1 + s_1 p_2 + p_3) = 3p_3 + bp_3$

$b(s_4 + s_3 p_1 + s_2 p_2 + s_1 p_3 + p_4) = 4p_4 + bp_4$

....

$b(s_{n-1} + s_{n-2} p_1 + s_{n-3} p_2 + s_{n-4} p_3 + ... + p_{n-1}) = (n-1)p_{n-1} + bp_{n-1}$

$b(s_n + s_{n-1} p_1 + s_{n-2} p_2 + s_{n-3} p_3 + ... + s_1 p_{n-1} + p_n) = np_n + bp_n$

$b(s_{n+1} + s_n p_1 + s_{n-1} p_2 + s_{n-2} p_3 + ... + s_2 p_{n-1} + s_1 p_n + p_{n+1}) =$
$(n+1)p_{n+1} + bp_{n+1}$

satisfed. From here using $|p_n| \leq 2$ inequality orderly; we can take the estimations for first five coefficients easily.

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Conference Proceedings

$B1$. Melike Aydogan; Presented a research paper. (Some Results On Janowski Starlike Log-Harmonic Mappings GFTA 2009 ROMAIN, SBU, Lucian Blaga University, 28 Agust-2 September , International Symposium on Geometric Function Theory and Applications,) General Mathematics Vol.17, No.4 (2009), 171-183.
$B2$. Melike Aydogan, Presented a research paper.(Distortion Properties of Janowski Starlike Log-Harmonic Mappings of Complex order b), Abstract ID:100346 AUS-ICMS Sharjah, UAE, Sharjah American University, American Mathematical Society, 18-21 March 2010, The First International Conference on Mathematics and Statistics

*B*3. Melike Aydogan, Presented a research paper. (Application of Subordination Principle to Log-Harmonic alpha-Spirallike Mappigs,), GFTA'2010 PROCEEDINGS VOLUME , page 146, 26 Agust-2 September Sofia, Bulgaria ,
*B*4. A. Yemisci, Y.Polatoglu and M. Aydogan, Presented a research paper. (Distortion Theorem and the radius of Convexity), GFTA'2011 Proceedings Vol, 4-8 September 2011,Babe Bolyai University, Romania.
*B*5. Y.Polatoglu, M. Aydogan and A.Yemisci, Presented a research paper. (Growth Theorem and the raidus of Close-to-Spirallike), Transform Methods and Special Functions 2011, 6 th International Conference , October 20-23 , Sofia-Bulgaria.

Invited Talks

$C1$. M.Aydogan, TMD, Sabanci University, Karakoy, Growth Theorem and the raidus of starlikeness of close-to-spirallike Functions, 22.04.2011,

National Publications

D1. Melike Aydogan, Some Results About Taxicab Geometry, Istanbul Aydin University, Fen Bilimleri Dergisi (September 2009) , Some Results On Taxicab Geometry, Year :1 ; Volume:1, ISSN:1309-1352

Papers which are submitted

$F1$. M. Aydogan, A. Sen, Y.Polatoglu, Applied Mathematics Letters , Two Point Distortion Theorem for Starlike Harmonic Mappings, 2011 (editor)
$F2$. M. Aydogan, A. Yemisci, Y.Polatoglu, Generalization of Close-to-Star Functions, Applied Mathematics and Computation, 2011, (editor)
$F3$. M. Aydogan, E. Yavuz Duman, S.Owa, Notes on Starlike Log-Harmonic Functions of order alpha, Bulletin of Mathematical Analysis and Applications, 2011 (editor)
$F4$. M. Aydogan, Notes on Janowski Starlike Log-Harmonic mappings of complex order b , The Bulletin of Korean Mathematical Society, (2012) (editor)
$F5$. M. Aydogan, Some Results on a Subclass of alpha-Spirallike Mappings, Applied and Computational Mathematics (SCI) (editor)

National Conferences and Seminars

1. Turkish Mathematical Society XVII. National Mathematics Symposium, August 23-26, 2004, Abant Izzet Baysal University, Bolu, Turkey
2. Turkish Mathematical Society XVIII. National Mathematics Symposium, September 05-08, 2005, Istanbul Kultur University, Istanbul, Turkey
3. Seminar series of Univalent and Geometric Function Theory, given by Shigeyoshi Owa, February 25-March 06, 2006, Istanbul Kultur University, Istanbul, Turkey
4. Summer School on Geometric Function Theory , .stanbul Kultur University, Agust 16- 25 2010
5. M. Aydogan, TMD, Karakoy Analysis Seminars, Growth Theorem and the raidus of starlikeness of close-to-spirallike Functions, 2011, Sabanci University, Karakoy

Courses Teached

Basic Mathematics, General Mathematics I, II; Calculus I, II; Analysis I, II; Introduction to Probability Theory and Statistics; Differential Equations I, Complex Analysis I, II; Statisitcs I, II.

Memberships

Turkish Mathematical Society 2004-pres.
American Mathematical Society 2009-pres
European Mathematical Society 2010-pres.
Yesilyurt Sports Club 1990-pres.
Classical Turkish Music Society of IKU 2004-pres.

Rewiever

1) Aip Advances
2) Applied Mathematics Letters
3) Computer and Mathematics with Applications
4) Applied Mathematics and Computation

Employment

Assistant Professor, Yeni Yuzyil University, Istanbul, Turkey 2010-pres.
Assistant Professor, Istanbul Aydin University, Istanbul, Turkey -2010
Instructor, Istanbul Aydin University, Istanbul, Turkey, 2005-2010
Mathematics Teacher, Bahcelievler High School, Istanbul, Turkey, 2001-2005

Education

Istanbul Kultur University, Istanbul, Turkey, Ph.D. in Mathematics
Thesis Title: Janowski Star like Log - Har monic Mapping of complex
2006-2009
Advisor: Yasar Polatoglu
Istanbul Kultur University, Istanbul, Turkey, M.Sc. in Computer and Mathematics, 2004-2006
Thesis title: Taxicab Geometry
Thesis advisor: Yasar Polatoglu
Istanbul University, Istanbul, Turkey , B.S. in Mathematics, 1997-2001
Sisli Terakki High School, Istanbul, Turkey, 1990-1997
Sisli 19 Mays Primary School, Istanbul, Turkey, 1985-1990

MY INTERNATIONAL REFERENCES

1) Prof. Shigeyoshi Owa, Kinki University, Kyoto-Japan, He has more than 600 papers in our field. Father of GFTA. mail: shige21@ican.zaq.ne.jp
2) Prof. Zayid Abdulhadi, American University of Sharjah, He first discovered log-harmonic mappings in 1984. mail: zahadi@aus.edu
3) Prof. Sanford Miller, Brockport College, Newyork, He is one of the famous ones who has more than 2000 papers and lots of theorems mail: smiller@brockport.edu
4) Prof. Grigore Salagean, University of Babes Bolyai, Romania, He is also very famous have hundreds of papers and HE HAS AN OPERATOR! mail: salagean@math.ubbcluj.ro
5) Prof. Virginia Kiryakova, Technical University of Sofia, Faculty of Applied Mathematics and Informatics, She is one of the main organizer of GFTA conferences. Mail: virginia@diogenes.bg
6) Prof. Daniel Breaz, Rector of 1 Decembrie 1918 University, Romania, mail: dbreaz@aub.ro

THANK YOU FOR YOUR PATIENCE!
Dr. Melike AYDOGAN