Projections of Fractals - Old and New

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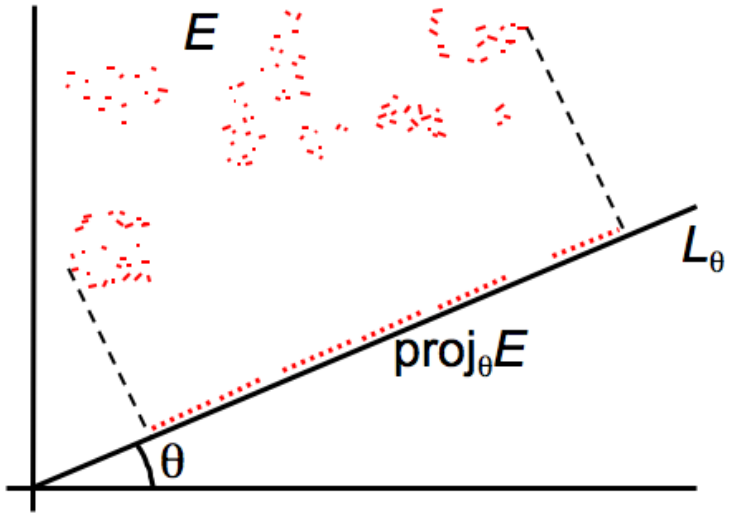
Marstrand's projection theorems



Theorem (Marstrand 1954) Let $E \subset \mathbb{R}^2$ be a Borel set.
For all $\theta \in [0, \pi)$

(i) $\dim_H \text{proj}_\theta E \leq \min\{\dim_H E, 1\}$.

For almost all $\theta \in [0, \pi)$,

(ii) $\dim_H \text{proj}_\theta E = \min\{\dim_H E, 1\}$,

(iii) $\mathcal{L}(\text{proj}_\theta E) > 0$ if $\dim_H E > 1$.

[proj_θ denotes orthogonal projection onto the line L_θ, $\dim_H$ is Hausdorff dimension, $\mathcal{L}$ is Lebsegue measure or length on L_θ.]

Energy characterisation of Hausdorff dimension

That $\dim_{\mathrm{H}} \mathrm{proj}_\theta E \le \min\{\dim_{\mathrm{H}} E, 1\}$ for all θ follows since projection is a Lipschitz map which cannot increase dimension.

Marstrand's lower bound proof was geometric and intricate. Kaufman's (1968) potential theoretic proof has become the standard approach for such problems.

This depends on the following energy characterisation of Hausdorff dimension. Let $\mathcal{M}(E)$ be the set of probability measures on E

$$\begin{aligned}\dim_{\mathrm{H}} &= \sup\left\{s : \frac{1}{C^{(s)}(E)} \equiv \inf_{\mu\in\mathcal{M}(E)} \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty\right\}\\ &= \sup\left\{s : C^{(s)}(E) > 0\right\}.\end{aligned}$$

Box-counting dimensions of projections

The box-counting dimension of a non-empty and compact $E \subset \mathbb{R}^2$ is

$$\dim_{\mathrm{B}} E \;=\; \lim_{r \to 0} \frac{\log N_r(E)}{-\log r}$$

where $N_r(E)$ is the least number of sets of diameter r covering E. [Taking lower/upper limits gives the lower/upper box dimensions.]

Is there a Marstrand-type theorem for box-dimensions of projections? For $E \subset \mathbb{R}^2$

$$\frac{\dim_{\mathrm{B}} E}{1 + \frac{1}{2}\dim_{\mathrm{B}} E} \leq \dim_{\mathrm{B}} \mathrm{proj}_\theta E \leq \min\{\dim_{\mathrm{B}} E, 1\} \quad \text{(almost all } \theta \in [0, \pi)\text{)}$$

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and examples show that these bounds are best possible.

Even so, $\underline{\dim}_B \mathrm{proj}_\theta E$ and $\overline{\dim}_B \mathrm{proj}_\theta E$ must be constant for almost all θ; for a messy argument and indirect value see (F & Howroyd, 1996, 2001). Using capacities things become much simpler.

Box-counting dimensions of projections

Define potential kernels $\phi_r^s(x)$ by

$$\phi_r^s(x) = \min\left\{1, \left(\frac{r}{|x|}\right)^s\right\} \qquad (x \in \mathbb{R}^2 \text{ or } \mathbb{R}^n)$$

The capacity $C_r^s(E)$ of a compact $E \subset \mathbb{R}^n$ w.r.t. ϕ_r^s is

$$\frac{1}{C_r^s(E)} = \inf_{\mu \in \mathcal{M}(E)} \int\int \phi_r^s(x-y)\,d\mu(x)\,d\mu(y),$$

where $\mathcal{M}(E)$ are the probability measures on E. The infimum is attained by some equilibrium measure $\mu_0 \in \mathcal{M}(E)$, and moreover

$$\int \phi_r^s(x-y)\,d\mu_0(y) \;\geq\; \frac{1}{C_r^s(E)} \quad (x \in E),$$

with equality for μ_0-almost all $x \in E$.

Box-counting dimensions of projections

Then for $E \subset \mathbb{R}^n$

$$c_1 C_r^s(E) \leq N_r(E) \leq \begin{cases} c_2 \log(1/r)\ C_r^s(E) & \text{if } s = n \\ c_2\ C_r^s(E) & \text{if } s > n \end{cases} \quad (1),$$

(c_1, c_2 depend on $n, s, \operatorname{diam} E$).

In particular for $E \subset \mathbb{R}^n$

$$\lim_{r \to 0} \frac{\log C_r^n(E)}{-\log r} = \lim_{r \to 0} \frac{\log N_r(E)}{-\log r} = \dim_{\mathrm{B}} E$$

(we can replace $\dim_{\mathrm{B}}$ and lim by either $\underline{\dim}_{\mathrm{B}}$ and $\underline{\lim}$, or by $\overline{\dim}_{\mathrm{B}}$ and $\overline{\lim}$).

Note: Inequalities (1) fail if $0 < s < n$.

Box-counting dimensions of projections

Theorem Let $E \subset \mathbb{R}^2$ be non-empty compact.
For all $\theta \in [0, \pi)$

$$\text{(i) } \dim_B \operatorname{proj}_\theta E \ \leq \ \lim_{r \to 0} \frac{\log C_r^1(E)}{-\log r}$$

For almost all $\theta \in [0, \pi)$,

$$\text{(ii) } \dim_B \operatorname{proj}_\theta E \ = \ \lim_{r \to 0} \frac{\log C_r^1(E)}{-\log r}$$

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Box-counting dimensions of projections

Theorem Let $E \subset \mathbb{R}^2$ be non-empty compact.
For all $\theta \in [0, \pi)$

$$\text{(i) } \dim_{\mathrm{B}} \operatorname{proj}_\theta E \;\le\; \lim_{r \to 0} \frac{\log C_r^1(E)}{-\log r} \;\equiv \dim_{\mathrm{B}}^1 E.$$

For almost all $\theta \in [0, \pi)$,

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[We can replace $\dim_{\mathrm{B}}$ and lim by either $\underline{\dim}_{\mathrm{B}}$ and $\underline{\lim}$, or by $\overline{\dim}_{\mathrm{B}}$ and $\overline{\lim}$.]

We call

$$\dim_{\mathrm{B}}^s E := \lim_{r \to 0} \frac{\log C_r^s(E)}{-\log r} \quad (E \subset \mathbb{R}^2 \text{ or } \mathbb{R}^n),$$

using capacity with respect to the kernel $\phi_r^s(x) = \min\left\{1, \left(\frac{r}{|x|}\right)^s\right\}$, the *s*-box-dimension profile of E, which should be thought of as the 'box-dimension of E when regarded from an s-dimensional viewpoint'.

Box-counting dimensions of projections

Lower bound proof: Let $F \subset \mathbb{R}$ be compact, ν a probability measure on F, and $\mathcal{I}_r(F)$ the intervals $[ir, (i+1)r), (i \in \mathbb{Z})$ that intersect F.

$$1 = \Big(\sum_{I \in \mathcal{I}_r(F)} \nu(I) \Big)^2 \leq N_r(F) \sum_{I \in \mathcal{I}_r(F)} \nu(I)^2 \leq$$

$$N_r(F) \sum_{I \in \mathcal{I}_r(F)} (\nu \times \nu)\{(w,z) \in I \times I\} \leq N_r(F)(\nu \times \nu)\{(w,z) : |w - z| \leq r\}. \quad (1)$$

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Let μ be an equilibrium measure on $E \subset \mathbb{R}^2$, with projections μ_θ onto L_θ.

$$\int (\mu_\theta \times \mu_\theta)\{(w,z) : |w-z| \leq r\} d\theta = \int (\mu \times \mu)\{(x,y) : |\mathrm{proj}_\theta x - \mathrm{proj}_\theta y| \leq r\} d\theta$$

$$= \iint \mathcal{L}\{\theta : |\mathrm{proj}_\theta (x-y)| \leq r\} d\mu(x) d\mu(y) \leq c \iint \phi_r^1(x-y) d\mu(x) d\mu(y) = \frac{c}{C_r^1(E)}.$$

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$$= \iint \mathcal{L}\{\theta : |\mathrm{proj}_\theta (x-y)| \leq r\} d\mu(x) d\mu(y) \leq c \iint \phi^1_r(x-y) d\mu(x) d\mu(y) = \frac{c}{C^1_r(E)}.$$

If $\sum_k 2^{sk} C^1_{2^{-k}}(E)^{-1} < \infty$ then there are $M_\theta < \infty$ for a.a. θ such that

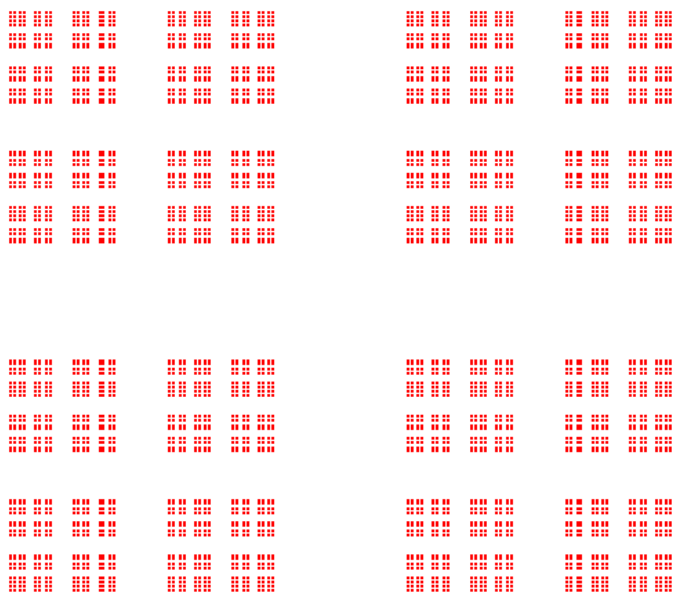
$$2^{sk}(\mu_\theta \times \mu_\theta)\{(w,z) : |w-z| \leq 2^{-k}\} \leq M_\theta \quad (k \in \mathbb{N}),$$

so from (1), $1 \leq N_{2^{-k}}(\mathrm{proj}_\theta E)\, 2^{-sk} M_\theta$.

Hence if $\overline{\dim}^1_{\mathrm{B}}(E) > s$ then $\overline{\dim}_{\mathrm{B}}(\mathrm{proj}_\theta E) \geq s$ for almost all θ.

Exceptional directions

Marstrand's theorem tells nothing about which particular directions may have projections with dimension or measure smaller than usual, i.e. when $\dim_{\mathrm{H}} \mathrm{proj}_\theta E < \min\{\dim_{\mathrm{H}} E, 1\}$, or if $\dim_{\mathrm{H}} E > 1$ when $\mathcal{L}(\mathrm{proj}_\theta E) = 0$.



The set shown has dimension $\log 4 / \log(5/2) = 1.51$, but with some projections of dimension < 1.

Exceptional directions

The set of exceptional directions can't be 'too big':

Theorem (Kaufman, 1968) If $E \subset \mathbb{R}^2$ and $\dim_{\mathrm{H}} E \leq 1$,

$$\dim_{\mathrm{H}}\{\theta : \dim_{\mathrm{H}} \mathrm{proj}_\theta E < \dim_{\mathrm{H}} E\} \leq \dim_{\mathrm{H}} E.$$

– follows from an energy estimate

Theorem (F, 1982) If $E \subseteq \mathbb{R}^2$ and $\dim_{\mathrm{H}} E > 1$,

$$\dim_{\mathrm{H}}\{\theta : \mathcal{L}(\mathrm{proj}_\theta E) = 0\} \leq 2 - \dim_{\mathrm{H}} E.$$

– proof uses Fourier transforms.

Theorem (F, Howroyd, 1997) For compact $E \subset \mathbb{R}^2$ and let $0 \leq s \leq 1$. Then

$$\dim_{\mathrm{H}}\{\theta \in [0, \pi) \text{ such that } \dim_{\mathrm{B}} \mathrm{proj}_\theta E < \dim_{\mathrm{B}}^s E\} \leq s.$$

General problem

Theorem (Marstrand 1954) Let $E \subset \mathbb{R}^2$ be a Borel set.
For all $\theta \in [0, \pi)$

(i) $\dim_{\mathrm{H}} \mathrm{proj}_\theta E \leq \min\{\dim_{\mathrm{H}} E, 1\}$.

For almost all $\theta \in [0, \pi)$,

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(iii) $\mathcal{L}(\mathrm{proj}_\theta E) > 0$ if $\dim_{\mathrm{H}} E > 1$.

General problem: find sets or classes of sets for which there are no exceptional directions for Marstrand's theorem, i.e. where (ii) or (iii) hold for all θ, or at least where the exceptional directions can be identified.

We will consider self-similar and self-affine sets and their random counterparts.

Self-similar sets

Given an iterated function system of contracting similarities $f_1, \ldots, f_m : \mathbb{R}^2 \to \mathbb{R}^2$ there exists a unique non-empty compact $E \subset \mathbb{R}^2$ called a self-similar set such that

$$E = \bigcup_{i=1}^{m} f_i(E). \qquad (*)$$

We assume that the union $(*)$ is disjoint or perhaps 'nearly disjoint' (i.e. OSC), so $\dim_{\mathrm{H}} E = \dim_{\mathrm{B}} E = s$, where s is the similarity dimension, given by $\sum_{i=1}^{m} r_i^s = 1$.

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Write the similarities as

$$f_i(x) = r_i O_i(x) + t_i$$

where $0 < r_i < 1$ is the scale factor, O_i is a rotation and t_i is a translation.

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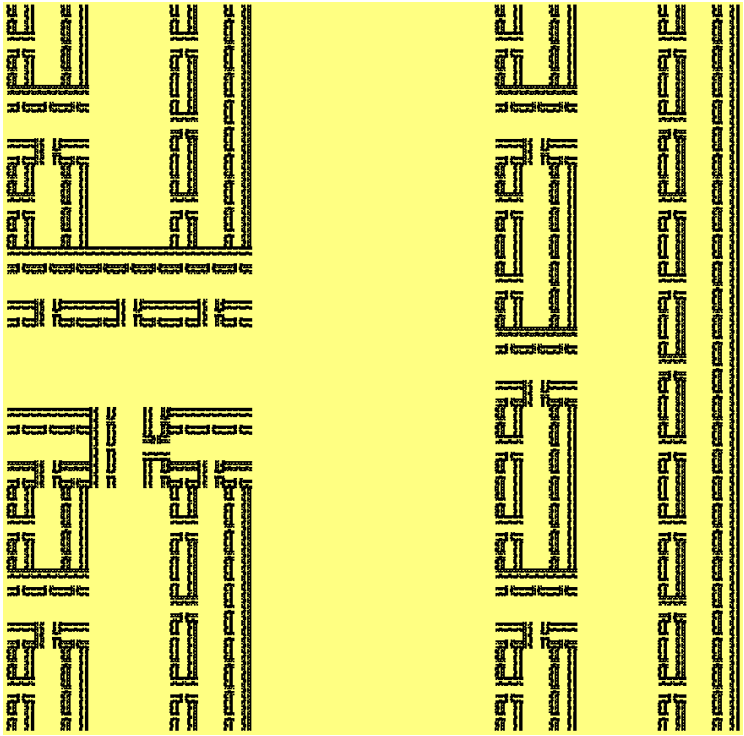
The family $\{f_1, \ldots, f_m\}$ has dense rotations if at least one of the O_i is a rotation by an irrational multiple of π, equivalently if group$\{O_1, \ldots, O_m\}$ is dense in $SO(2, \mathbb{R})$.
Otherwise $\{f_1, \ldots, f_m\}$ has finite rotations.



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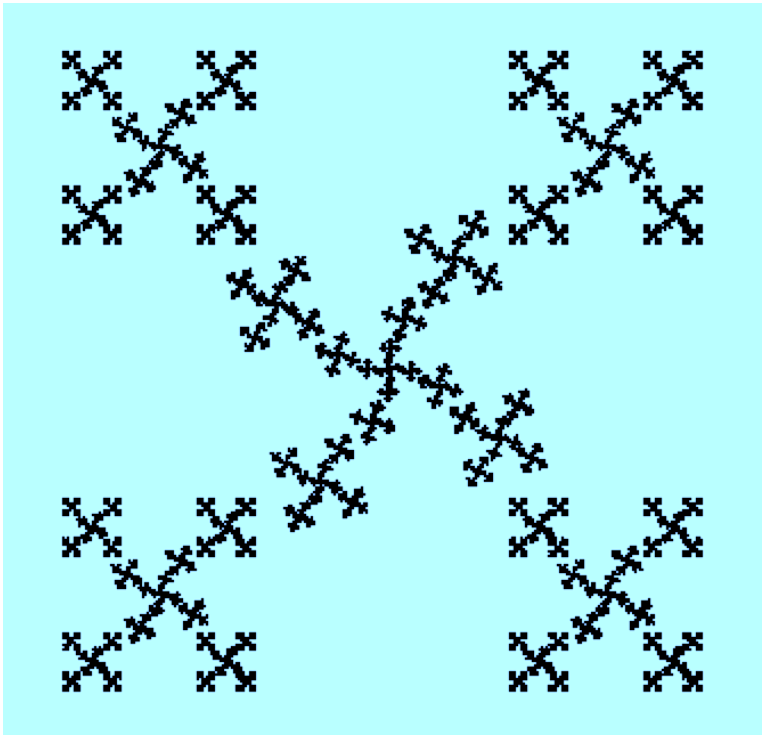

dense rotations

Self-similar sets

Self-similar sets with finite rotations

Theorem (Farkas 2014) Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with finite rotations and satisfying OSC. Then there is at least one value of θ such that $\dim_{\mathrm{H}} \mathrm{proj}_\theta E < \dim_{\mathrm{H}} E$

Self-similar sets with dense rotations

Theorem (Peres & Shmerkin 2009, Hochman & Shmerkin 2012)
Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with dense rotations. Then

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Theorem (Shmerkin & Solomyak 2014) Let $E \subset \mathbb{R}^2$ be the self-similar attractor of an IFS with dense rotations with $1 < \dim_{\mathrm{H}} E < 2$. Then $\mathcal{L}(\mathrm{proj}_\theta E) > 0$ for all θ except (perhaps) for a set of θ of Hausdorff dimension 0.

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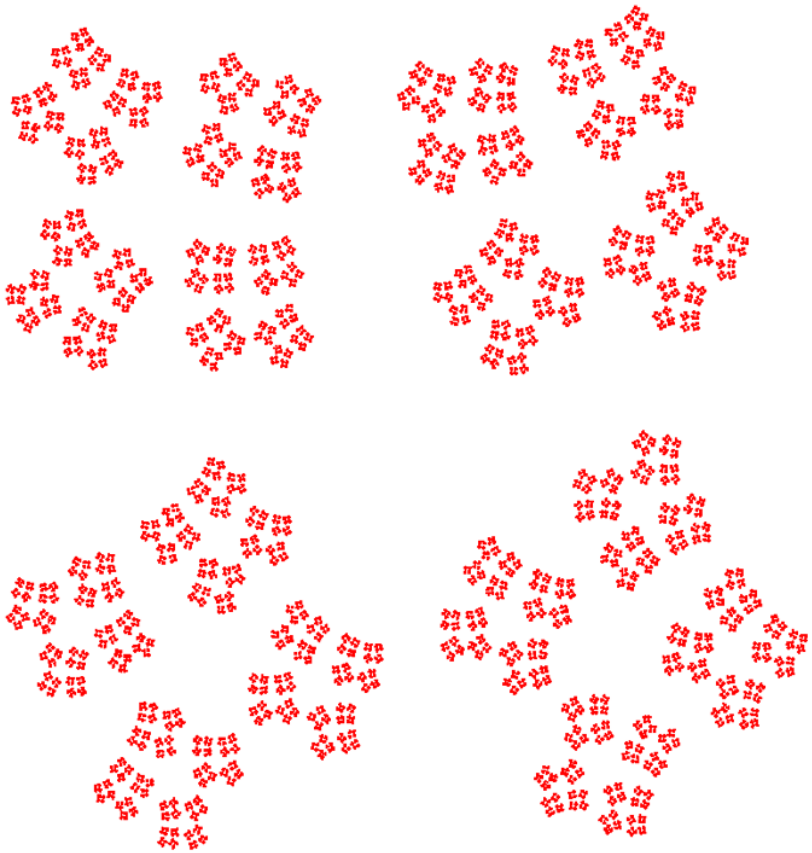
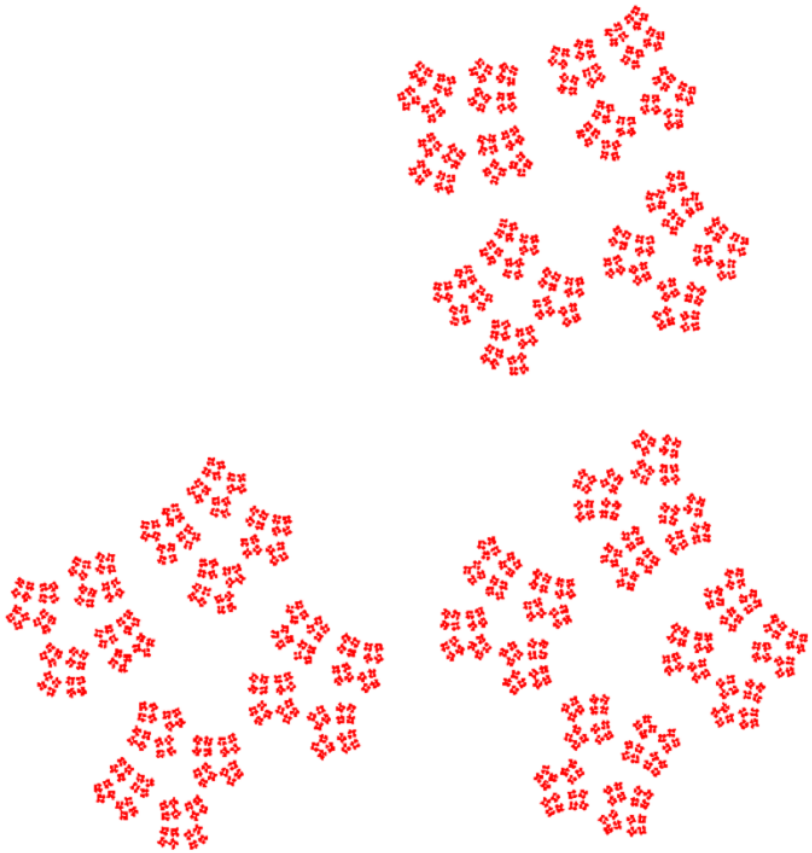
Theorem (Rapaport 2017) There exists a self-similar $E \subset \mathbb{R}^2$ defined by an IFS $\{f_i(x) = rO(x) + t_i\}_{i=1}^4$ satisfying the SSC with O an irrational rotation, such that $\dim_{\mathrm{H}} E > 1$ but $\mathcal{L}(\mathrm{proj}_\theta E) = 0$ for $\theta \in S$ for a dense G_δ set $S \subset [0, \pi)$. (O is a rotation by a reciprocal of a pisot number).

Percolation on a self-similar set



Start with a self similar set. At each stage of the iterated construction, retain each component with probability p.

Percolation on a self-similar set



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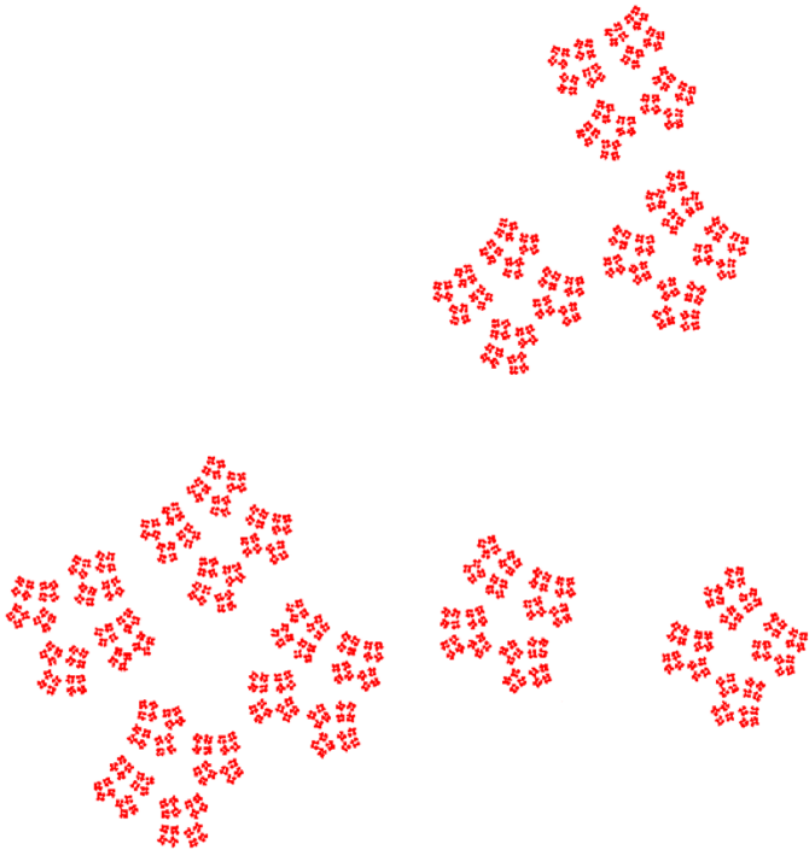

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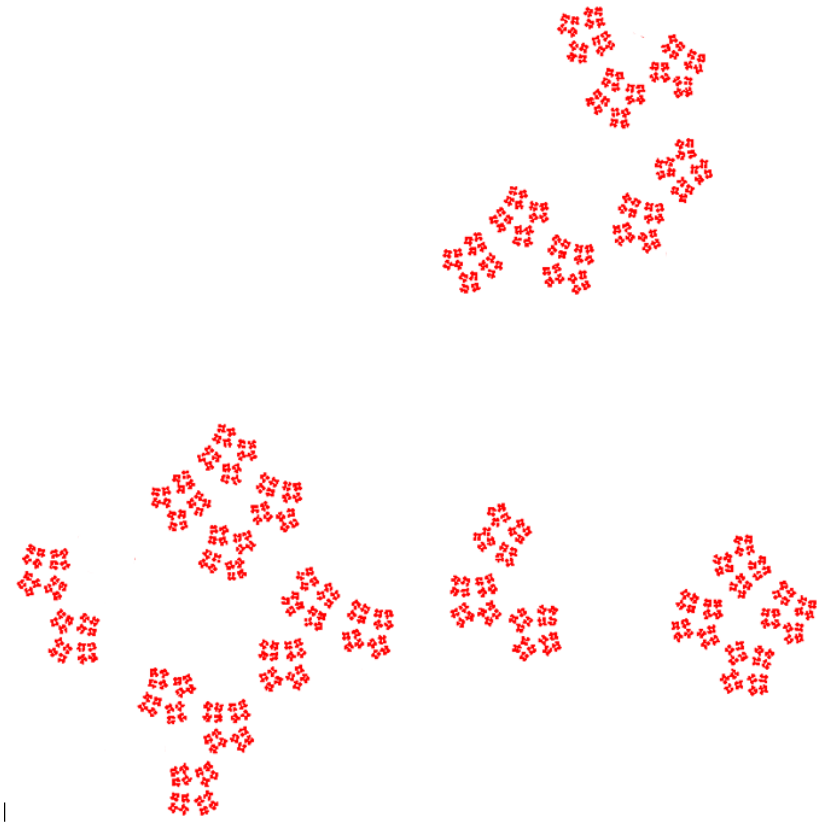

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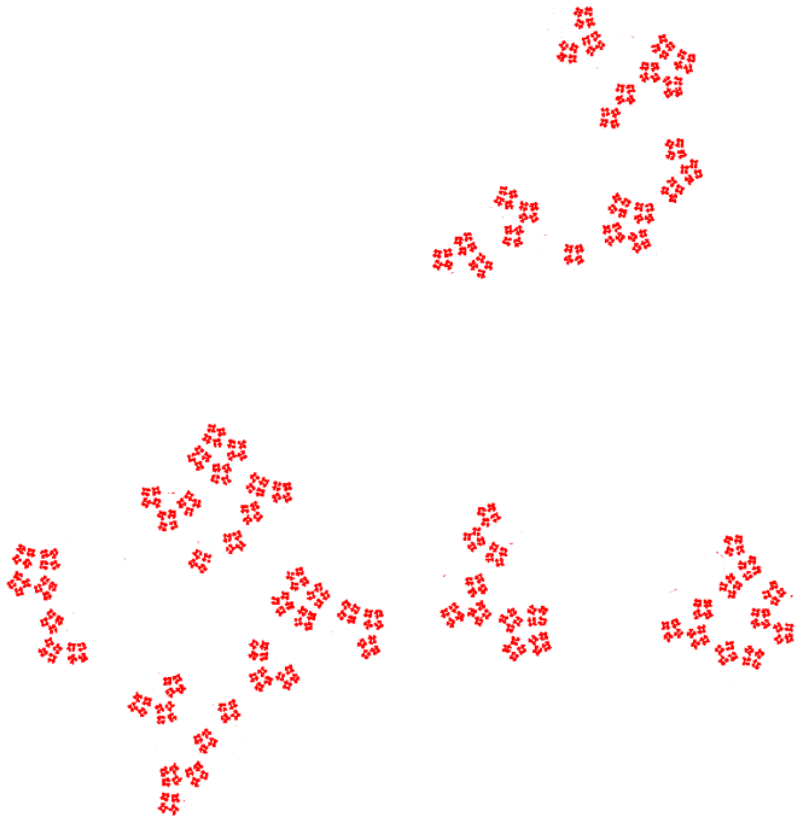

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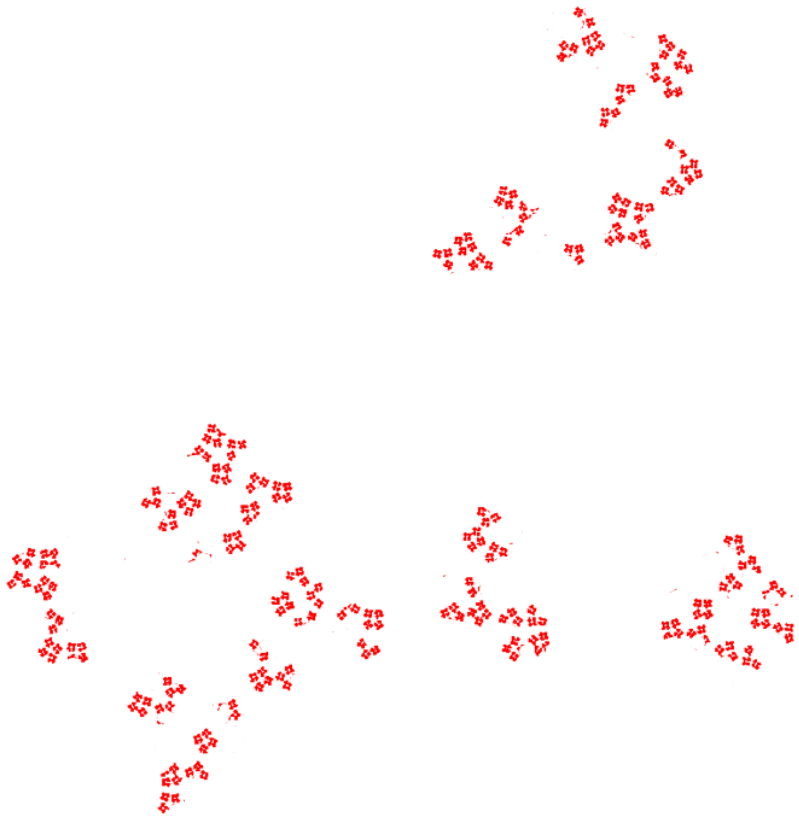

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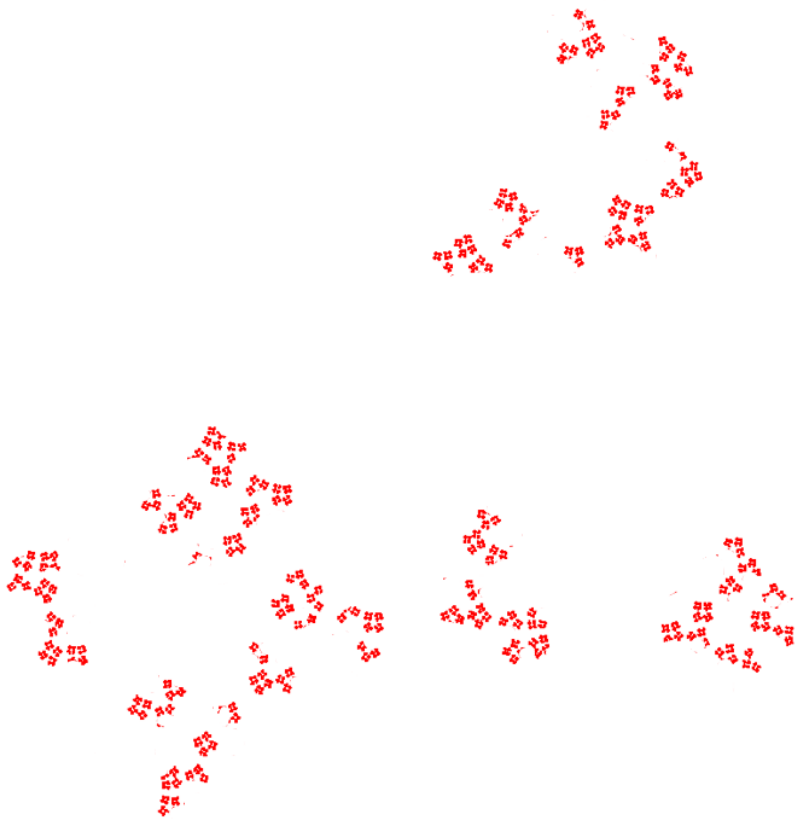

If $p > 1/m$ then $E_p \neq \emptyset$ with positive probability, conditional on which $\dim_{\mathrm{H}} E_p = s$, where $p \sum_{i=1}^{m} r_i^s = 1$

Projection of percolation on a self-similar set

Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with dense rotations. Perform the percolation process on E with respect to its hierarchical construction, with each similar component retained independently with probability $p > 1/m$, to obtain a percolation set $E_p \subset E$. Let s be given by $p \sum_{i=1}^{m} r_i^s = 1$, so, conditional on $E_p \neq \emptyset$, $\dim_{\mathrm{H}} = s$ almost surely.

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Theorem (Jin & F, 2014) Conditional on $E_p \neq \emptyset$, almost surely

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta E_p = \min\{\dim_{\mathrm{H}} E_p, 1\} \text{ for all } \theta.$$

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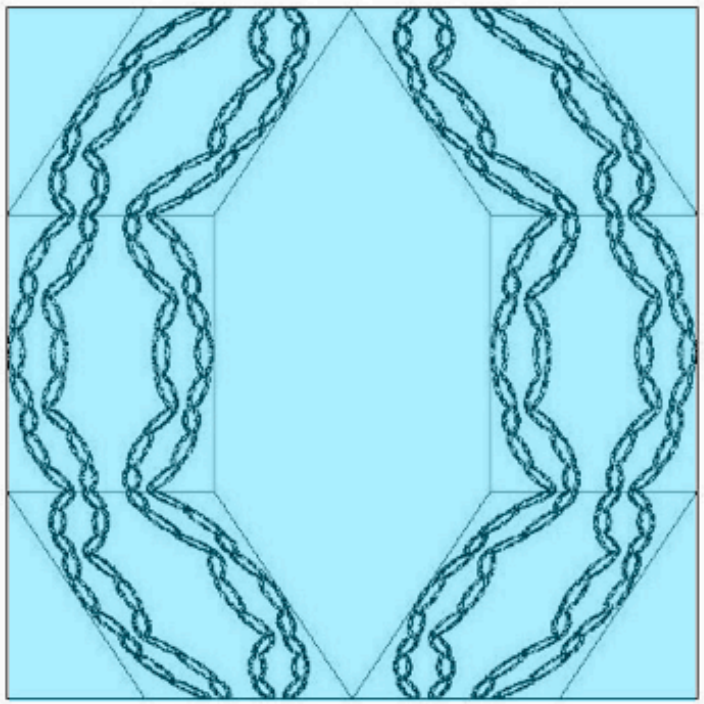
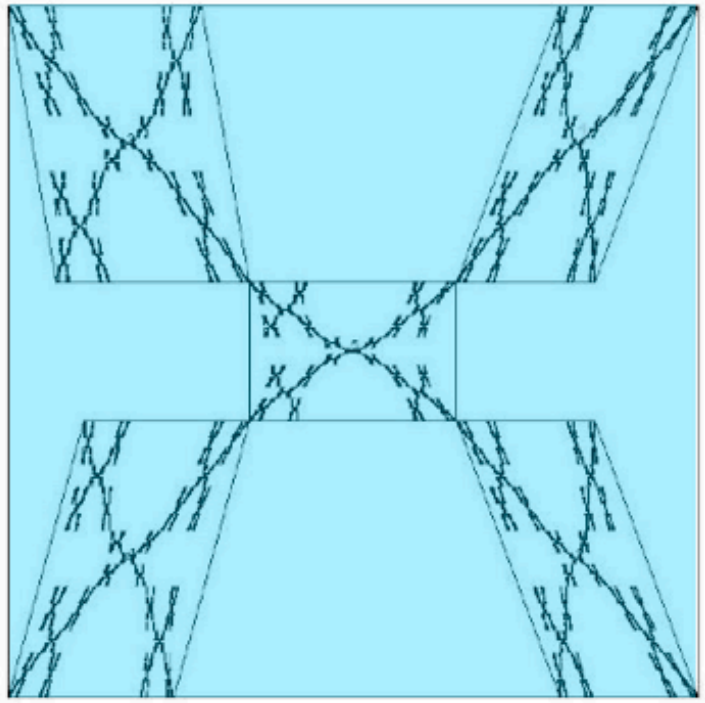
Theorem (Jin & F, 2015) If $s > 1$, then conditional on $E_p \neq \emptyset$, almost surely $\mathcal{L}(\mathrm{proj}_\theta E_p) > 0$ for all θ except for a set of θ of Hausdorff dimension 0.

[These results on projections of random percolating sets have implications for sections of deterministic sets.]

Self-affine sets

Now let the S_i be *affine* contractions, i.e. $f_i(x) = T_i(x) + t_i$ where T_i is a linear mapping and t_i is a translation.





Some self-affine sets

Self-affine sets

The singular values $\alpha_1 \geq \alpha_2 \geq 0$ of a linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ are the $+$ve square roots of the eigenvalues of TT^*.
The singular value function of T is

$$\phi^s(T) = \begin{cases} \alpha_1^s & (0 \leq s \leq 1) \\ \alpha_1 \alpha_2^{s-1} & (1 \leq s) \end{cases}$$

The affinity dimension of the self-affine set E defined by the IFS $\{f_i(x) = T_i(x) + t_i\}_{i=1}^m$ is

$$\dim_{\mathrm{Aff}} E = \Big\{ s : \lim_{k \to \infty} \Big(\sum_{i_1 \ldots i_k} \phi^s(T_{i_1} \circ \cdots \circ T_{i_k}) \Big)^{1/k} = 1 \Big\}.$$

In particular, if $\|T_i\| < \frac{1}{2}$ for all i, then $\dim_{\mathrm{H}} E = \dim_{\mathrm{Aff}} E$ for almost all $(t_1, \ldots, t_m) \in \mathbb{R}^{2m}$.

Projections of self-affine sets

A Bernoulli measure μ on a self-affine E is of the form $\mu(T_{i_1} \circ \cdots \circ T_{i_k} E) = p_{i_1} \cdots p_{i_k}$ where $(p_1, \ldots, p_m)$ is a given probability vector.

Theorem (Kempton & F, 2016) If E is a self-affine set defined by strictly positive matrices and there exists a sequence (μ_n) of Bernoulli measures supported on E with $\dim_{\mathrm{H}} \mu_n \to \dim_{\mathrm{Aff}} E$, then $\dim_{\mathrm{H}} E = \dim_{\mathrm{Aff}} E$ and

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta(E) = \min\{\dim_{\mathrm{H}} E, 1\} \quad \text{for all } \theta \in [0, \pi)$$

unless the T_i have a common maximal eigenvector when there may be one exceptional direction.

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unless the T_i have a common maximal eigenvector when there may be one exceptional direction.

Corollary (Kempton & F, 2016) If T_i are positive matrices with $\|T_i\| < \frac{1}{2}$ for all i with no common maximal eigenvector, then for almost all $(t_1, \ldots, t_m) \in \mathbb{R}^{2m}$, $\dim_{\mathrm{H}} E = \dim_{\mathrm{Aff}} E$ and

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta(E) = \min\{\dim_{\mathrm{H}} E, 1\} \quad \text{for all } \theta \in [0, \pi).$$

However, this corollary refers to 'generic' self-affine sets, and to find specific E to which the theorem can be applied can be awkward.

Projections of self-affine sets

Very recently Morris & Shmerkin showed that all projections behave well for a large class of identifiable self-affine sets E.

Theorem (Morris & Shmerkin, 2017) Suppose that the IFSs satisfy SOSC, the T_i have no common eigenvector, one of the T_i is hyperbolic, and the T_i have 'exponential separation'. If either

$$\dim_{\mathrm{Aff}} E \geq \tfrac{3}{2} \quad \text{or} \quad \alpha_1(T_i)^2 \leq \alpha_2(T_i) \text{ for all } i$$

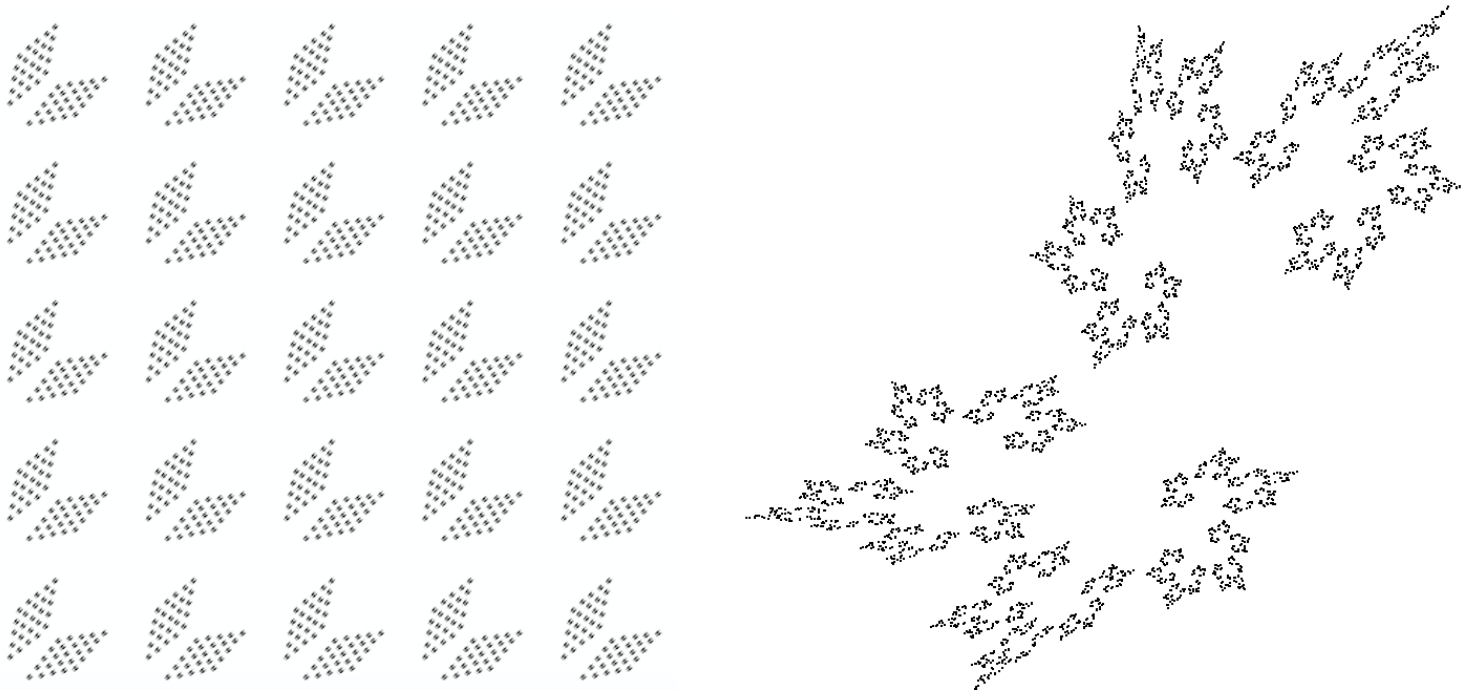
then

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{B}} E = \dim_{\mathrm{Aff}} E$$

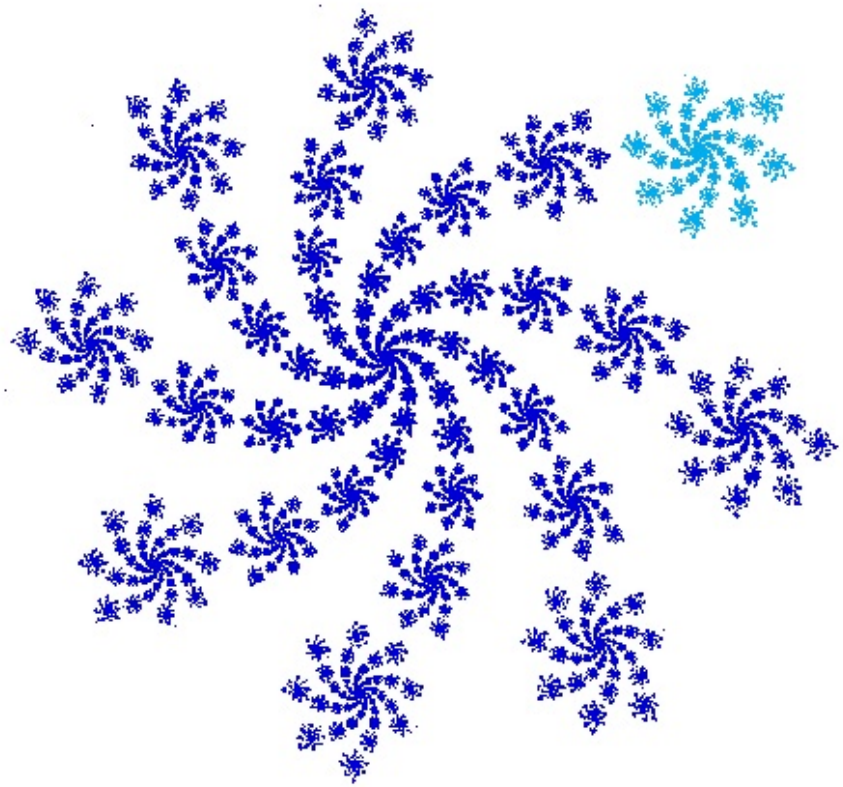
and

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta(E) = \min\{\dim_{\mathrm{H}} E, 1\} \quad \text{for all } \theta \in [0, \pi).$$

The proof depends on showing that given $\epsilon > 0$ there exists $k \in \mathbb{N}$ and a Bernoulli measure μ with respect to $\{T_{i_1} \circ \cdots \circ T_{i_k}\}$ supported on the self-affine set E such that $\dim_H \mu > \dim_{\mathrm{Aff}} E - \epsilon$ so that the previous Corollary can be applied.



Specific self-affine sets arising from 50 and 2 IFS mappings with all projections of Hausdorff dimension 1



Thank you!