# On a recursive construction of Dirichlet form on the Sierpiński gasket

Qingsong Gu
(Joint work with Ka-Sing Lau and Hua Qiu)

The Chinese University of Hong Kong

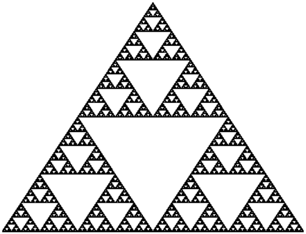
*qsgu@math.cuhk.edu.hk*

June 2017

# Background

Recall that a **Sierpiński gasket** is the unique non-empty compact set $K$ in $\mathbb{R}^2$ satisfying $K = \bigcup_{i=1}^{3} F_i(K)$ for an iterated function system (IFS) $\{F_i\}_{i=1}^{3}$ on $\mathbb{R}^2$ such that $F_i(x) = \frac{1}{2}(x - p_i) + p_i$ with $p_1 = 0,\ p_2 = 1,\ p_3 = \exp\left(\frac{\pi\sqrt{-1}}{3}\right)$.

Denote by $V_0 = \{p_1, p_2, p_3\}$ the **boundary** of $K$, and let $F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}$ for a word $\omega \in W_n = \{1,2,3\}^n$. Let $V_n = \bigcup_{\omega \in W_n} F_\omega(V_0)$ and $V_* = \bigcup_{n=0}^{\infty} V_n$.



# Background

1. $K = \overline{V_*}$ under the Euclidean metric.
2. The Hausdorff dimension of $K$ is $\alpha = \frac{\log 3}{\log 2}$.
3. The **standard Dirichlet form** $(\mathcal{E}, \mathcal{F})$ on $L^2(K.\nu)$ is well-known:

$$\mathcal{E}(u) = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n \sum_{p\sim_n q} (u(p) - u(q))^2, \tag{1}$$

$$\mathcal{F} = \{u \in C(K) : \ \mathcal{E}(u) < \infty\} \tag{2}$$

where $p \sim_n q$ means $p \neq q$ and $p, q \in F_\omega(V_0)$ for some word $\omega \in W_n$.

4. **Self-similar identity**:

$$\mathcal{E}(u) = \sum_{i=1}^{3} \frac{1}{r_i} \mathcal{E}(u \circ F_i), \tag{3}$$

where $r_i > 0$, $i = 1, 2, 3$ are called **renormalization factors**.

# Background

**How about Dirichlet forms without self-similar identity?**

For $n \geq 0$, let $\Gamma_n$ be the graph on $V_n$ with edge relation $\sim_n$, for $u \in \ell(V_n)$, let $(\mathcal{E}_n, \ell(V_n))$ be

$$\mathcal{E}_n(u) = \sum_{p \sim_n q} c_{pq}^{(n)} (u(p) - u(q))^2,$$

where $c_{pq}^{(n)} \geq 0$ are **conductances**.

**Compatible**: the restriction of $\mathcal{E}_n$ on $\ell(V_{n-1})$ must be $\mathcal{E}_{n-1}$.

In [Meyers,Strichartz,Teplyaev 2004], the authors use the compatible condition to solve the equations of conductances from a given harmonic function.

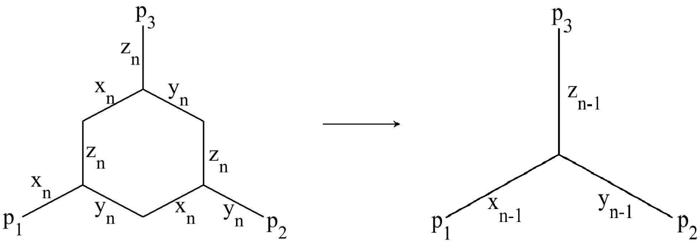
Now let us consider a special case, that is we require the conductances of the cells $F_\omega(V_0)$ on the same level $|\omega| = n$ are the same. We call this kind of construction **the recursive construction**.

# Background

Let $(a_0, b_0, c_0)$ be the conductance on $V_0$, and let $(a_n, b_n, c_n)$ be the conductances of $F_\omega(V_0), |\omega| = n, n \geq 1$ to be determined. $\Delta - Y$ transform gives $(x_n, y_n, z_n)$ on the $Y$-side. Compatibility:

$$\begin{cases} x_{n-1} = x_n + \phi(x_n; y_n, z_n), \\ y_{n-1} = y_n + \phi(y_n; z_n, x_n), \\ z_{n-1} = z_n + \phi(z_n; x_n, y_n), \end{cases} \qquad n \geq 1, \tag{4}$$

where $\phi(x_n; y_n, z_n) := \frac{(x_n+y_n)(x_n+z_n)}{2(x_n+y_n+z_n)}$.



# Construction

## Proposition

*For $a_0, b_0, c_0 > 0$, in order that (4) to have positive solution $(x_n, y_n, z_n), n \geq 1$, it is necessary and sufficient that $x_0 \geq y_0 = z_0 > 0$ (or the symmetric alternates).*
*In this case, $x_n \geq y_n = z_n > 0,\ n \geq 0$ and $\{(x_n, y_n, z_n)\}_{n \geq 0}$ is uniquely determined by the initial data $(x_0, y_0, z_0)$.*

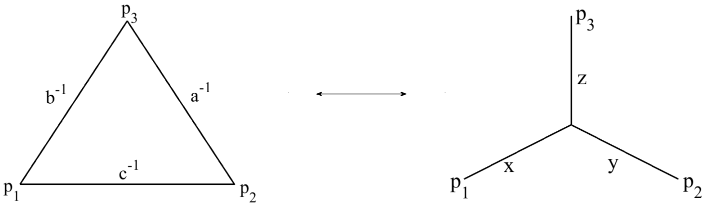



Figure: Consistence of the n-th and (n − 1)-th resistance networks

# Construction

**Sketch of the Proof.**

*Sufficiency*.

$$\begin{cases} x_1 = \frac{1}{15}\left(14x_0 + 3y_0 - 2\sqrt{4x_0^2 + 6x_0y_0 + 6y_0^2}\right), \\ y_1 (= z_1) = \frac{1}{5}\left(-2x_0 + y_0 + \sqrt{4x_0^2 + 6x_0y_0 + 6y_0^2}\right), \end{cases} \tag{5}$$

*Necessity*. Assume that $x_0 \geq y_0 > z_0$. We show that $x_n \gg y_n \gg z_n$ and from this we deduce that $z_n < 0$, a contradiction.

# Construction

Recall that the **(effective) resistance** $R := R^{(a_0,b_0)}$ on $V_* \times V_*$ is defined for any two distinct points $x, y \in V_*$,

$$R(x,y)^{-1} := \inf\{\mathcal{E}(u) : \ u \in \ell(V_*), \ u(x) = 1, u(y) = 0\}.$$

## Proposition

*For $a_0 > b_0 = c_0$, the completion of the $(V_*, R^{(a_0,b_0)})$ is $K$, and*

$$C^{-1}|x-y| \le R^{(a_0,b_0)}(x,y) \le C|x-y|^{\gamma'}, \qquad x, y \in K$$

*where $\gamma' = \frac{\log 3}{\log 2} - 1$ and $C > 0$ is a constant depends on $a_0$ and $b_0$. Furthermore $R^{(a_0,b_0)}$ is a bounded metric with*

$$\sup\left\{R^{(a_0,b_0)}(x,y) : x, y \in K\right\} \le C' b_0^{-1}. \tag{6}$$

*where $C' > 0$ is independent of $a_0$ and $b_0$.*

# Construction

## Theorem

*For the case* $x_0 > y_0 = z_0 > 0$ *in the above theorem, we have*

$$a_n = \frac{x_n}{y_n(2x_n + y_n)} \asymp 2^n, \qquad b_n = c_n = \frac{1}{2x_n + y_n} \asymp \left(\frac{3}{2}\right)^n.$$

*Moreover* $\mathcal{E}^{(a_0,b_0)}(u) = \lim_{n\to\infty} \mathcal{E}_n^{(a_0,b_0)}(u)$ *defines a strongly local regular Dirichlet form on* $L^2(K,\mu)$*; it satisfies*

$$\mathcal{E}^{(a_0,b_0)}(u) = \sum_{i=1}^{3} \mathcal{E}^{(a_1,b_1)}(u \circ F_i) \tag{7}$$

*but does* **NOT** *satisfy the energy self-similar identity.*

# Construction

The **dichotomic** situation:

1. case 1: $x_0 = y_0$. the standard Dirichlet form[J.Kigami, Analysis on fractals].
2. case 2: $x_0 > y_0$. First created by [K.Hattori,T.Hattori,H.Watanabe1994], they call it the asymptotically one-dimensional diffusion processes on the SG (later studied by [B.Hambly,T.Kumagai1996], [B.Hambly,O.Jones 2002], [B.Hambly,W.Yang arXiv1612.02342]).

# Spectrum asymptotic

For the Dirichlet form $(\mathcal{E}^{(a_0,b_0)},\mathcal{F})$ with $a_0>b_0$, we study the distribution of the eigenvalues. Let $\Delta^{(a_0,b_0)}$ be the **Laplacian**, the infinitesimal generator of $\left(\mathcal{E}^{(a_0,b_0)},\mathcal{F}\right)$ on $L^2(K,\mu)$, where $\mu$ is fixed to be the normalized $\alpha$-Hausdorff measure. Denote by $\rho^{(a_0,b_0)}(t)$ the eigenvalue count of the **Dirichlet boundary condition** (D.B.C), that is

$$\rho^{(a_0,b_0)}(t)=\#\left\{\lambda\le t:\lambda \text{ is an eigenvalue of } -\Delta^{(a_0,b_0)} \text{ with D.B.C.}\right\}, \tag{8}$$

and denote by $\rho_N^{(a_0,b_0)}(t)$ the count with **Neumann boundary condition** (N.B.C).

# Spectrum asymptotic

Our main result is the following.

## Theorem

*Assume that $a_0 > b_0 = c_0$, then*

$$\rho^{(a_0,b_0)}(t) \asymp t^{\frac{\log 3}{\log(9/2)}}, \qquad t \to \infty.$$

This estimate improves the lower bound of $\rho^{(a_0,b_0)}(t)$ in [B.Hambly,O.Jones2002,Theorem 13] where it was shown to be $C^{-1}t^{\log 3/\log(9/2)}(\log t)^{-\beta}$ with $\beta > \log 3/\log 2$, using a heat kernel technique in the estimation.

# Spectrum asymptotic

The first basic lemma:( bounds of the $\lambda_1$)

## Lemma

*There exists $C > 0$ such that for any initial data $a > b = c > 0$ on $\Gamma_0$, we have*

$$C^{-1}b \leq \lambda_1^{(a,b)} \leq Cb, \tag{9}$$

*where $\lambda_1^{(a,b)}$ is the first eigenvalues of $-\Delta^{(a,b)}$ with the Dirichlet boundary condition.*

# Spectrum asymptotic

**Sketch of proof**

We will make use of the **Rayleigh quotient** for the first eigenvalue:

$$\lambda_1 = \inf_{u \in \mathcal{F}_0, u \neq 0} \frac{\mathcal{E}(u)}{||u||_2^2}, \tag{10}$$

where $\mathcal{F}_0 := \{u \in \mathcal{F} : u|_{V_0} = 0\}$.

# Spectrum asymptotic

The 1-harmonic function $u_1$ gives the upper bound of $\lambda_1$,

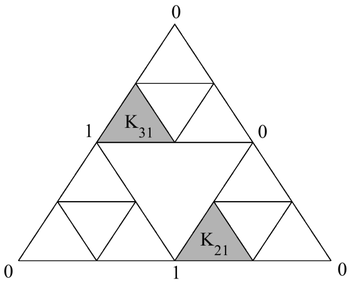



Figure: the value of $u_1$

and the uniform upper bound of $R^{(a,b)}$ gives the lower bound of $\lambda_1$.

# Spectrum asymptotic

The Second lemma:( scaling property)

**Lemma**

*Let* $a_0 > b_0 = c_0$*, then for all* $t \geq 0$ *and* $n \geq 0$*,*

$$3^n \rho^{(a_n,b_n)}\left(\frac{t}{3^n}\right) \leq \rho^{(a_0,b_0)}(t), \quad \text{and} \quad \rho_N^{(a_0,b_0)}(t) \leq 3^n \rho_N^{(a_n,b_n)}\left(\frac{t}{3^n}\right). \tag{11}$$

For the idea, we refer to the similar proof in [J.Kigami,M.Lapidus1993,Propositions 6.2, 6.3], in which, they use the self-similar identity (3). Here we use the identity (7).

# Spectrum asymptotic

The third lemma:(dimension of the first eigenfunction space)

**Lemma**

*Let $K$ be the Sierpiński gasket and $\mu$ be the normalized Hausdorff measure on $K$. Let $(\mathcal{E}^{(a,b)}, \mathcal{F})$ be the Dirichlet form defined in Theorem 1.1. Let $\Lambda_1$ be the eigenfunction space of $\lambda_1$, the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. Then $\Lambda_1$ is of dimension one.*

The idea of this lemma comes essentially from [E.Davies One-parameter semigroups, Theorems 7.2, 7.3].

# Spectrum asymptotic

**Proof of Theorem 3 by using the above three lemmas.**
By Lemma 6, $\rho^{(a_n,b_n)}\left(\lambda_1^{(a_n,b_n)}\right)=1$, and $\rho_N^{(a_n,b_n)}\left(\lambda_1^{(a_n,b_n)}\right)\le\rho\left(\lambda_1^{(a_n,b_n)}\right)+3=4.$ then by Lemma 5, we have

$$\rho_N^{(a_0,b_0)}\left(3^n\lambda_1^{(a_n,b_n)}\right)\le 4\cdot 3^n.$$

Letting $t=3^n\lambda_1^{(a_n,b_n)}$, by Lemma 4, we have $t\asymp 3^n b_n\asymp 3^n(3/2)^n=(9/2)^n$ and $3^n\asymp t^{\log 3/\log(9/2)}$. It follows that

$$\rho^{(a_0,b_0)}(t)\le\rho_N^{(a_0,b_0)}(t)\le Ct^{\frac{\log 3}{\log(9/2)}}$$

for some $C>0$. The same argument yields the other inequality.

# Other examples

As shown in [B.Hambly,T.Kumagai1998] as examples, Dirichlet forms of the second kind exist on higher dimensional, higher level Sierpiński gaskets and also the Vicsek sets.

Here we would like to give some other interesting examples which have different results by using the recursive construction.

The first one is a modification of the Sierpiński gasket, we define the *twisted Sierpiński gasket* to be the unique nonempty compact set $K$ on $\mathbb{R}^2$ with the contractions $\{T_i\}_{i=1}^3$ such that

$T_1(x) = \overline{\frac{x-p_1}{2}} \cdot \exp\left(\frac{\pi\sqrt{-1}}{3}\right) + p_1, \;\; T_2(x) = \overline{\frac{x-p_2}{2}} \cdot \exp\left(-\frac{\pi\sqrt{-1}}{3}\right) + p_2,$

and $T_3(x) = -\frac{x-p_3}{2} + p_3$, (i.e., $T_i$ reflects the sub-triangle $K_i$ along the angle bisection at $p_i$).

# Other examples

the compatibility of $\{(x_n, y_n, z_n)\}_{n\geq 0}$ must satisfy the following equations:

$$\begin{cases} x_{n-1} = x_n + \psi(x_n; y_n, z_n), \\ y_{n-1} = y_n + \psi(y_n; z_n, x_n), \\ z_{n-1} = z_n + \psi(z_n; x_n, y_n), \end{cases} \qquad n \geq 1. \tag{12}$$

where $\psi(x_n; y_n, z_n) = \frac{2y_n z_n}{x_n+y_n+z_n}$.

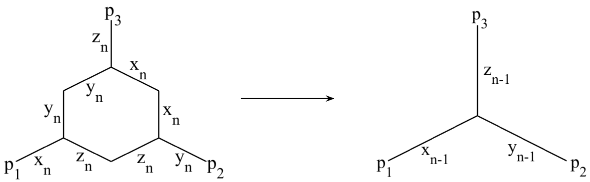



Figure: Δ-Y transform for the twisted maps

# Other examples

## Proposition

*For $x_0, y_0, z_0 > 0$, in order that $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ in (12) have positive solution $(x_n, y_n, z_n)$, $n \geq 1$, it is necessary and sufficient that $x_0 \geq y_0 = z_0 > 0$ (or the symmetric alternates). In this case, $\{(x_n, y_n, z_n)\}_{n=0}^{\infty}$ is uniquely determined by $(x_0, y_0, z_0)$.*
*Furthermore, for $x_0 > y_0 = z_0$, we have the estimate $x_n \asymp 1, y_n = z_n \asymp \left(\frac{1}{3}\right)^n$, and hence $a_n \asymp 3^n$, $b_n = c_n \asymp 1$.*

Conclusion: The recursive construction only gives the standard Dirichlet form because in the case $x_0 > y_0 = z_0$, the resistance metric on $V_*$ is not comparable with the Euclidean metric and hence $\overline{V_*}^{R^{(a_0,b_0)}}$ is not homeomorphic to $K$.

# Other examples

The second example is the *Vicsek eyebolted cross* [G. and Lau arXiv1703.07061]:

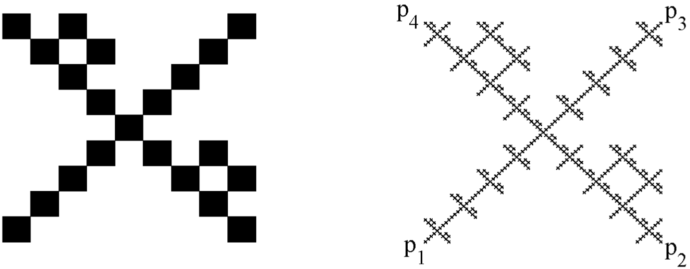
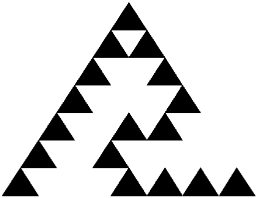



Figure: The Vicsek eyebolted cross

Conclusion: On the Vicsek eyebolted cross, the recursive construction cannot give a "standard" Dirichlet form but will give one of the second kind.

# Other examples

The third example is the *Sierpiński sickle* [G. and Lau arXiv1703.07061]:



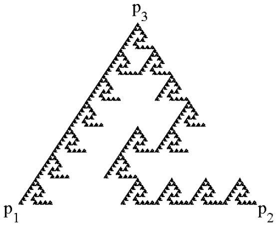



Figure: The Sierpiński sickle

Conclusion: On the Sierpiński sickle, the recursive construction gives nothing.

# Thank You !!