On the Complex Dimensions of Nonarchimedean Fractal Sets

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Definitions & Notation

Definitions & Notation: Homogeneous measures

Let (X, d, μ) be a complete, separable metric measure space such that

$$0 < \mu(B(x, r)) < \infty$$

for all $x \in X$ and $r > 0$. Let $A \subseteq X$.

Definition

We say that μ is **q-homogeneous** on A if there is some constant $M > 0$ such that

$$\frac{\mu(B(x, r))}{\mu(B(\xi, \rho))} \leq M \left(\frac{r}{\rho}\right)^q$$

for all $0 < \rho < r \leq \operatorname{diam}(A)$, all $x \in A$, and all $\xi \in B(x, r)$. The **measure theoretic Assouad dimension** of A is

$$\dim_{\mathrm{As}}(A) := \inf \{ q \geq 0 \,|\, \mu \text{ is } q\text{-homogeneous on } A \}.$$

Definitions & Notation: The distance zeta function

Definition

Suppose that $\dim_{\mathrm{As}}(X) = Q$ and that A is a bounded subset of X. For $\delta > 0$, define

$$A_\delta := \{x \in X \,|\, d(x, A) \leq \delta\}\,.$$

The **distance zeta function** associated to A is given by

$$\zeta_A(s) = \zeta_{A,A_\delta}(s) := \int_{A_\delta} d(x, A)^{s-Q}\, \mathrm{d}\mu(x)$$

Under relatively mild hypotheses on A, the integral above will diverge at—but be absolutely convergent to the right of—the upper Minkowski dimension of A.

Definition

Suppose that $\zeta_A(s)$ can be meromorphically extended to a (strictly) larger domain. Then the **complex dimensions** of A, denoted $\mathscr{P}(A)$, are the poles of this extension. That is

$$\mathscr{P}(A) := \{\omega \in \mathbb{C} \,|\, \omega \text{ is a pole of } \zeta_A(s)\}\,.$$

Definitions & Notation: p-adic spaces

Let p be a fixed prime number.

Definition

Let $r \in \mathbb{Q}$. The **p-adic absolute value** of r is given by

$$|r|_p := p^{-n},$$

where n is the unique integer such that there are $a, b \in \mathbb{Z}$ relatively prime to p with $r = p^n \frac{a}{b}$.

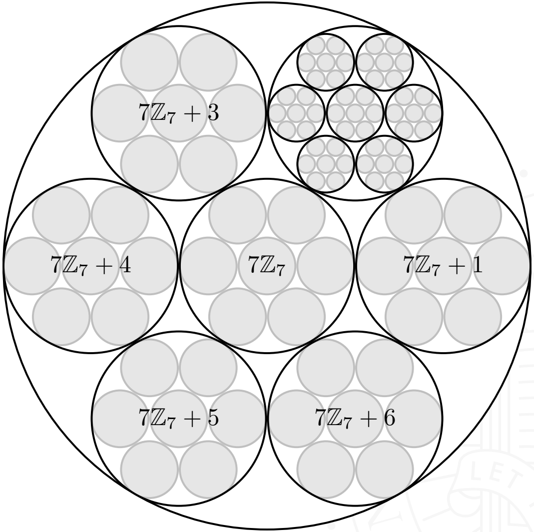
Definition

The **p-adic numbers**, denoted $\mathbb{Q}_p$, are the metric completion of $\mathbb{Q}$ with respect to the metric induced by the p-adic abolute value. The **p-adic integers**, denoted $\mathbb{Z}_p$, are elements of the "dressed" unit ball in $\mathbb{Q}_p$, i.e.

$$\mathbb{Z}_p := B_{\leq}(0,1) = \{x \in \mathbb{Q}_p \,|\, |x|_p \leq 1\}.$$

$\mathbb{Q}_p$ is equipped with the Haar measure μ such that $\mu(\mathbb{Z}_p) = 1$.

Definitions & Notation: p-adic spaces



Definitions & Notation: p-adic spaces

Let $Q \in \mathbb{N}$ and $\alpha \in [1, \infty)$.

Notation

On the product space $\mathbb{Q}_p^Q$, define the equivalent metrics

$$d^\alpha(\boldsymbol{x}, \boldsymbol{y}) := \left(\sum_{i=1}^{Q} |x_i - y_i|_p^\alpha \right)^{1/\alpha},$$

and

$$d^\infty(\boldsymbol{x}, \boldsymbol{y}) := \max \left\{ |x_i - y_i|_p \,\middle|\, 1 \leq i \leq Q \right\}.$$

Lemma

For any $Q \in \mathbb{N}$ and any $\alpha \in [1, \infty]$, the product space $(\mathbb{Q}_p^Q, d^\alpha, \mu)$ satisfies

$$\dim_{\mathrm{As}}(\mathbb{Q}_p^Q) = Q,$$

where μ is the natural product measure.

Definitions & Notation: Iterated function systems on $\mathbb{Q}_p^Q$

Definition

A **self-similar iterated function system** (SSIFS) on $\mathbb{Q}_p^Q$ is a finite collection of maps $\{\varphi_j\}_{j\in\mathscr{J}}$, each of which is of the form

$$\varphi_j(x) = p^{k_j}x + b_j,$$

where $k_j \in \mathbb{N}$ and $b_j \in \mathbb{Q}_p^Q$. We call p^{-k_j} the **contraction ratio** of φ_j. We associate to an SSIFS the map of sets

$$\Phi(E) := \bigcup_{j\in\mathscr{J}} \varphi_j(E).$$

Theorem

Let Φ be as above. Then there is a unique, nonempty, compact set $\mathscr{A} \subseteq \mathbb{Q}_p^Q$ such that

$$\Phi(\mathscr{A}) = \mathscr{A}.$$

We call $\mathscr{A}$ the ***attractor*** *of the SSIFS.*

Definitions & Notation: Iterated function systems on $\mathbb{Q}_p^Q$

Let $\{\varphi_j\}_{j\in\mathscr{J}}$ be an SSIFS.

Notation

Let $\mathscr{J}^*$ denote the set of all finite sequences (or "words") with entries in $\mathscr{J}$. For each

$$J = (j_1, j_2, \ldots, j_n) \in \mathscr{J},$$

define

$$\varphi_J = \varphi_{j_n} \circ \varphi_{j_{n-1}} \circ \cdots \circ \varphi_1.$$

Let $\omega = (\,) \in \mathscr{J}^*$ denote the "empty word." We adopt the convention that φ_ω is the identity map, i.e.

$$\varphi_\omega(x) = x.$$

Results & Examples

Results & Examples: Self-similar sets

Theorem

Let $\mathscr{A}$ *be the attractor of the SSIFS* $\{\varphi_j\}_{j\in\mathscr{J}}$ *on* $\mathbb{Q}_p^Q$*. Further suppose that* $b_j \in \mathbb{Z}_p$ *for each* j*, and that* $\varphi_j(\mathbb{Z}_p) \cap \varphi_{j'}(\mathbb{Z}_p) = \emptyset$ *for all* $j \neq j'$*. Then*

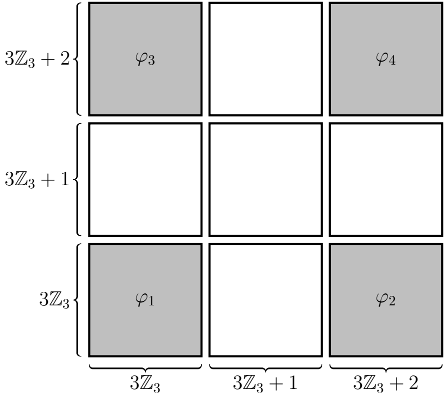
$$\zeta_{\mathscr{A}}(s) = \zeta_{\mathscr{A},\Omega_\iota}(s) \sum_{n=0}^{\infty} C_n p^{-ns},$$

where

$$\zeta_{\mathscr{A},\Omega_\iota}(s) = \int_{\mathbb{Z}_p^Q \setminus \Phi(\mathbb{Z}_p^Q)} d(x,\mathscr{A})^{s-Q}\,\mathrm{d}\mu(x),$$

and C_n *counts the number of maps of the form* φ_J *for some* $J \in \mathscr{J}^*$ *with contraction ratio* p^{-n}*.*

Results & Examples: 3-adic Cantor dust



Example

Let $\{\varphi_j\}_{j=1}^4$ be the SSIFS on $\mathbb{Q}_3^2$ that maps $\mathbb{Z}_3^2$ into the four rectangles shown to the left. Let $\mathscr{C}^2$ denote the attractor of this SSIFS.

We may also regard $\mathscr{C}^2$ as the Cartesian product of two copies of a 3-adic Cantor set. In either case, $\mathscr{C}^2$ is an analog of the ternary Cantor dust in $\mathbb{R}^2$.

Results & Examples: 3-adic Cantor dust

Example (con't)

With respect to d^∞*, we have*

$$\zeta_{\mathscr{C}^2,\Omega_\iota}(s) = \int_{\mathbb{Z}_3^2 \setminus \Phi(\mathbb{Z}_3^2)} d^\infty(x,\mathscr{C}^2)^{s-2}\,\mathrm{d}\mu(x) = \mu\left(\mathbb{Z}_3^2 \setminus \Phi(\mathbb{Z}_3^2)\right) = \frac{5}{9}.$$

Next, observe that

$$C_n := \#\{J \in \mathscr{J}^* \mid \varphi_J(x) = 3^n x + b_J\} = 4^n.$$

Hence

$$\zeta_{\mathscr{C}^2}(s) = \zeta_{\mathscr{C}^2,\Omega_\iota}(s)\sum_{n=0}^{\infty} C_n 3^{-ns} = \frac{5}{9}\sum_{n=0}^{\infty}\left(\frac{4}{3^s}\right)^n = \frac{5}{9}\frac{3^s}{3^s-4}.$$

Therefore

$$\mathscr{P}(\mathscr{C}^2) = \frac{\log(4)}{\log(3)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)}.$$

Results & Examples: Fibonacci attractors

Example

Fix a prime p and define maps on $\mathbb{Q}_p$ by

$$\varphi_1(x) = px, \qquad \textit{and} \qquad \varphi_2(x) = p^2x + 1.$$

Let $\mathscr{F}$ denote the attractor of the SSIFS $\{\varphi_1, \varphi_2\}$. Then

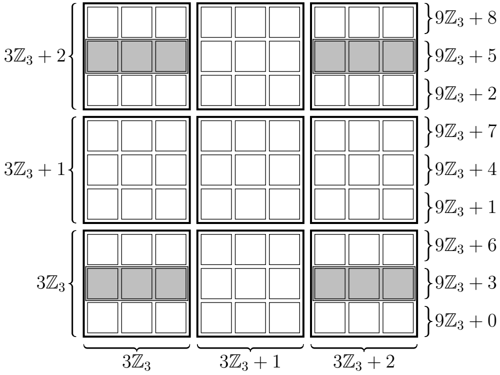
$$\mathscr{P}(\mathscr{F}) = \left(\frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(p)}\right) \cup \left(-\frac{\log(\phi)}{\log(p)} + \mathbb{i}\frac{(2\pi+1)\mathbb{Z}}{\log(p)}\right),$$

where

$$\phi = \frac{1+\sqrt{5}}{2}.$$

This recovers the result of Lapidus and Lũ' (2008), obtained in the setting of one-dimensional p-adic fractal strings. The current approach provides a broader context for the study of p-adic fractal strings, and avoids several technical difficulties.

Results & Examples: A simple McMullen carpet analog



Example

Let $\mathscr{A}$ denote the attractor of the IFS shown to the left. With respect to d^∞,

$$\mathscr{P}(\mathscr{A}) = \left(\frac{3\log(2)}{2\log(3)} + \mathbb{i}\frac{\pi\mathbb{Z}}{\log(3)} \right) \cup \left(\frac{\log(4)}{\log(3)} - 1 + \mathbb{i}\frac{2\pi\mathbb{Z}}{\log(3)} \right).$$

Selected Bibliography

[1] Michel L. Lapidus and Hùng Lũ', *Nonarchimedean cantor set and string*, J. Fixed Point Theory and Appl. **3** (2008), no. 1, 181–190.

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[3] Curt McMullen, *The Hausdorff dimension of general Sierpński carpets*, Nagoya Mathematical J. **96** (1984), 1–9.