Sierpiński carpet as a Martin boundary

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table of contents

Previous approaches to a Laplacian

graph approximation: define a discrete Laplacian on each step and get the Laplacian in the limit (Kigami (1989/93), Kusuoka/Zhou (1992), Strichartz (2001))

Brownian motion: the Laplace operator is the infinitesimal generator of the Brownian motion (i.e. Barlow/Bass (1989/99), Lindstrøm (1990))

function spaces: using the theory for function spaces, it is possible, to define differential operators like the Laplacian (Triebel (1997))

Martin boundary: see next slides (Denker/Sato (2001/02), Ju/Lau/Wang (2011))

Martin boundary theory

Let $(X_n)_{n\geq 1}$ be a Markov chain with state space $\mathcal{W}$ and transition probability $p(v,w)$ with $v,w\in\mathcal{W}$

Based on $p(v,w)$ define the Martin kernel $k(v,w)$ of $(X_n)_{n\geq 0}$

Define the Martin space $\overline{\mathcal{W}}$ by ρ-completion of $\mathcal{W}$ (ρ is a certain metric on $\mathcal{W}$ depending on k)

$\mathcal{M} := \partial\overline{\mathcal{W}} = \overline{\mathcal{W}}\backslash\mathcal{W}$ is called the Martin boundary

By DYNKIN's theorem (1969) exists for every non-negative p-harmonic function h on $\mathcal{W}$ a measure $\mu_h(\mathrm{d}\alpha)\geq 0$ on $\mathcal{M}$ such that

$$h(w) = \int_{\mathcal{M}} k(w,\xi)\mu_h(\mathrm{d}\alpha) \qquad \text{for all } w\in\mathcal{W}$$

holds.

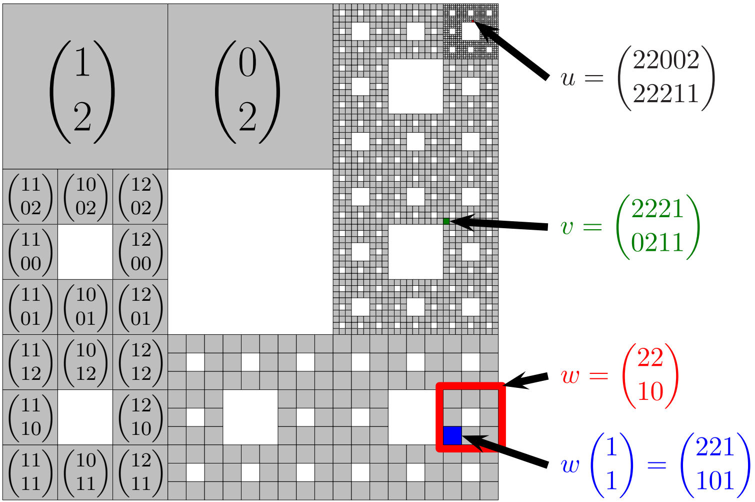
Sierpiński carpet as a Martin boundary

Define the alphabet

$$\mathcal{A} := \left\{ \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix} \right\}$$

$\begin{pmatrix}1\\2\end{pmatrix}$	$\begin{pmatrix}0\\2\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix}$		$\begin{pmatrix}2\\0\end{pmatrix}$
$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\end{pmatrix}$

Define the word space $\mathcal{W} := \bigcup_{n=1}^{\infty} \mathcal{A}^n \cup \{\emptyset\}$ ($\emptyset$ is the empty word)



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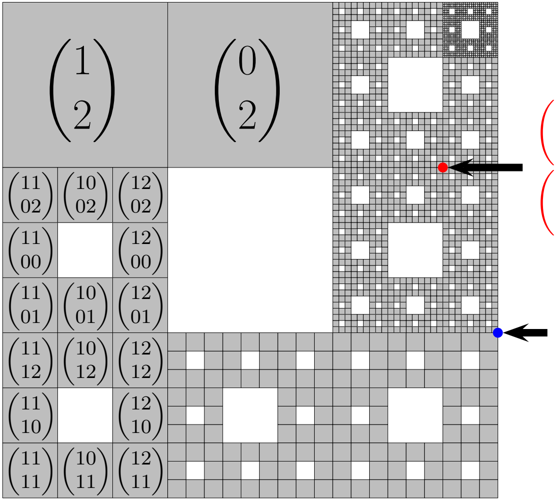


$$\begin{pmatrix} 202^\infty \\ 211^\infty \end{pmatrix} = \begin{pmatrix} 221^\infty \\ 211^\infty \end{pmatrix} = \begin{pmatrix} 202^\infty \\ 022^\infty \end{pmatrix} = \begin{pmatrix} 221^\infty \\ 022^\infty \end{pmatrix}$$

$$\begin{pmatrix} 22^\infty \\ 01^\infty \end{pmatrix} = \begin{pmatrix} 22^\infty \\ 12^\infty \end{pmatrix}$$

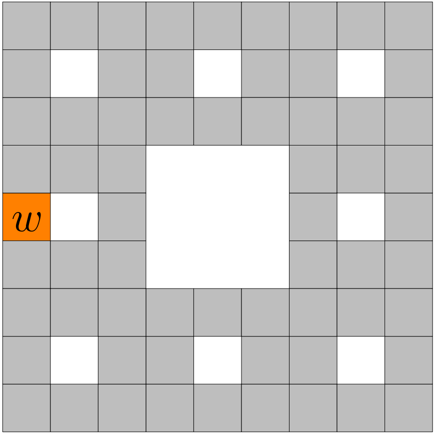
Define an equivalence relation $\sim$ on $\mathcal{W} \cup \mathcal{A}^\infty$:

1. let $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathcal{W} \cup \mathcal{A}^\infty$. Define $\overline{w_i}$ by:

   $$\overline{w_i} := \begin{cases} u(a+b)(2b)^k & \text{if } w_i = uab^k \text{ or } w_i = uab^\infty \text{ with} \\ & u \in \{0,1,2\}^n, a, b \in \{0,1,2\}, a \neq b, k \geq 1 \\ w_i & \text{else} \end{cases}$$

   where the addition/multiplication is $\mod 3$.
   Remark: it holds by definition, that $\overline{\overline{w_i}} = w_i$
2. set $\hat{w} := \begin{pmatrix} \overline{w_1} \\ w_2 \end{pmatrix}$, $\check{w} := \begin{pmatrix} w_1 \\ \overline{w_2} \end{pmatrix}$ and $\tilde{w} := \begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix}$
3. define the equivalence class of w by
   $[w] := \{\hat{w}, \check{w}, \tilde{w}\} \cap (\mathcal{W} \cup \mathcal{A}^\infty)$

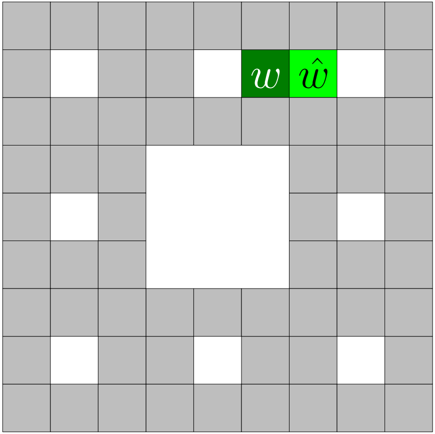
illustration of the equivalence relation $\sim$



$$w = \begin{pmatrix} 11 \\ 00 \end{pmatrix}$$

$\hat{w}$, $\check{w}$ and $\tilde{w}$ are equal to w

illustration of the equivalence relation $\sim$



$w = \begin{pmatrix} 02 \\ 20 \end{pmatrix}$,

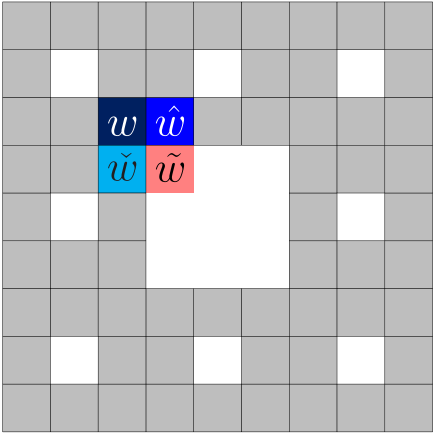
$\hat{w} = \begin{pmatrix} \overline{02} \\ 20 \end{pmatrix} = \begin{pmatrix} 21 \\ 20 \end{pmatrix}$

$\check{w}$ and $\tilde{w}$ don't distinguish, because

$$\check{w} = \begin{pmatrix} 02 \\ \overline{20} \end{pmatrix} = \begin{pmatrix} 02 \\ 20 \end{pmatrix} = w$$

$$\tilde{w} = \begin{pmatrix} \overline{02} \\ \overline{20} \end{pmatrix} = \begin{pmatrix} 21 \\ 20 \end{pmatrix} = \hat{w}$$

illustration of the equivalence relation $\sim$



$$w = \begin{pmatrix} 12 \\ 21 \end{pmatrix},$$

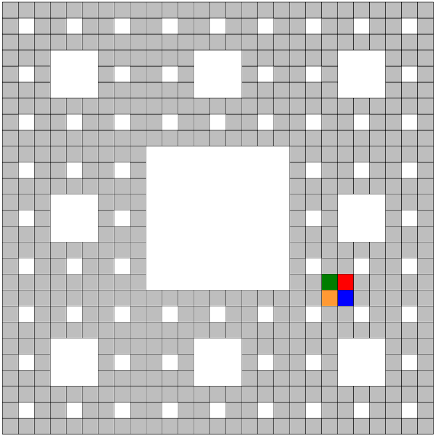
$$\hat{w} = \begin{pmatrix} \overline{12} \\ 21 \end{pmatrix} = \begin{pmatrix} 01 \\ 21 \end{pmatrix},$$

$$\check{w} = \begin{pmatrix} 12 \\ \overline{21} \end{pmatrix} = \begin{pmatrix} 12 \\ 02 \end{pmatrix}$$

$\tilde{w}$ doesn't exist, because

$$\tilde{w} = \begin{pmatrix} \overline{12} \\ \overline{21} \end{pmatrix} = \begin{pmatrix} 01 \\ 02 \end{pmatrix} \notin \mathcal{W}$$

illustration of the equivalence relation $\sim$



$$w = \begin{pmatrix} 201 \\ 011 \end{pmatrix},$$

$$\hat{w} = \begin{pmatrix} \overline{201} \\ 011 \end{pmatrix} = \begin{pmatrix} 212 \\ 011 \end{pmatrix},$$

$$\check{w} = \begin{pmatrix} 201 \\ \overline{011} \end{pmatrix} = \begin{pmatrix} 201 \\ 122 \end{pmatrix},$$

$$\tilde{w} = \begin{pmatrix} \overline{201} \\ \overline{011} \end{pmatrix} = \begin{pmatrix} 212 \\ 122 \end{pmatrix}$$

Set $R(w) := \#[w] = \#\{w^* \in \mathcal{W} : w^* \sim w\} \in \{1, 2, 3, 4\}$

Define the transition probability $p : \mathcal{W} \times \mathcal{W} \to [0, 1]$ by

$$p(v, w) := \begin{cases} \frac{1}{8 \cdot R(v)} & \text{if } \exists i \in \mathcal{A} \text{ and } v^* \sim v \text{ s.t. } w = v^* i, \\ 0 & \text{else} \end{cases}$$

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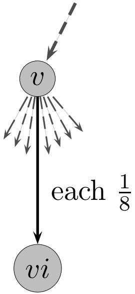
for example:
$R(v) = 1$:



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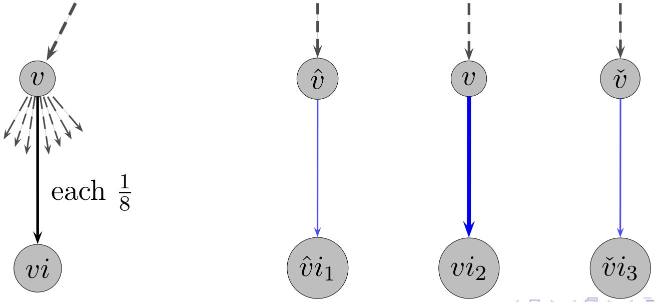

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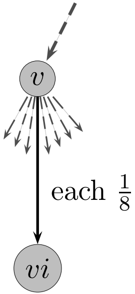

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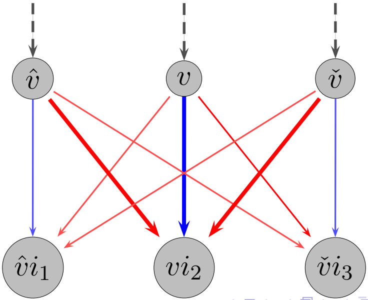

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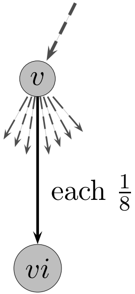
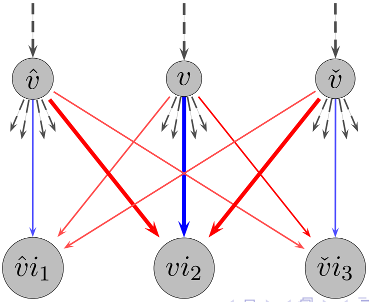


$R(v) = 3:$



each with probability $\frac{1}{8 \cdot 3} = \frac{1}{24}$

the way it goes on and a short outlook

p defines a Markov chain $(X_n)_{n\geq 0}$.

Futher define the n-step probability, the Green function, the Martin kernel, a metric ρ on $\mathcal{W}$ and the Martin boundary $\mathcal{M} = \overline{\mathcal{W}} \backslash \mathcal{W}$.

Denote by K the Sierpiński carpet, which is generated by the IFS $\{\mathbb{R}^2; S_1, \ldots, S_8\}$ and which fulfills $K = \bigcup_{i=1}^{8} S_i(K)$

Our aim is to prove, that

$$K \overset{?}{\cong} \mathcal{A}^{\infty}/_{\sim} \overset{?}{\cong} \mathcal{M}$$

holds.

Problems: choice of the metric on $\mathcal{A}^{\infty}/_{\sim}$

harmonic functions on the Sierpiński carpet

First hard question: What is the boundary of the Sierpiński carpet? And why?

11 22	10 22	12 22	01 22	00 22	02 22	21 22	20 22	22 22
11 20		12 20	01 20		02 20	21 20		22 20
11 21	10 21	12 21	01 21	00 21	02 21	21 21	20 21	22 21
11 02	10 02	12 02				21 02	20 02	22 02
11 00		12 00				21 00		22 00
11 01	10 01	12 01				21 01	20 01	22 01
11 12	10 12	12 12	01 12	00 12	02 12	21 12	20 12	22 12
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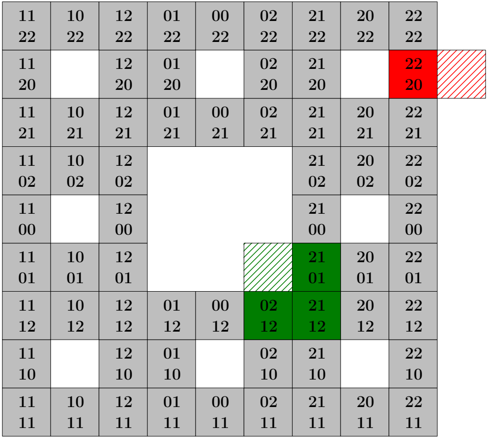

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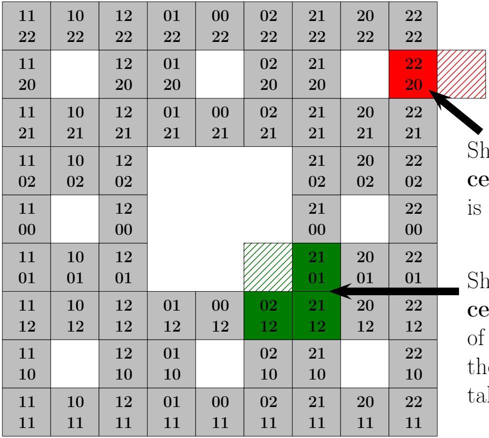

Should be a **boundary cell**, because the cell is out!

Should be **no boundary cells**, because the absence of the cell comes from the structure of the fractal!

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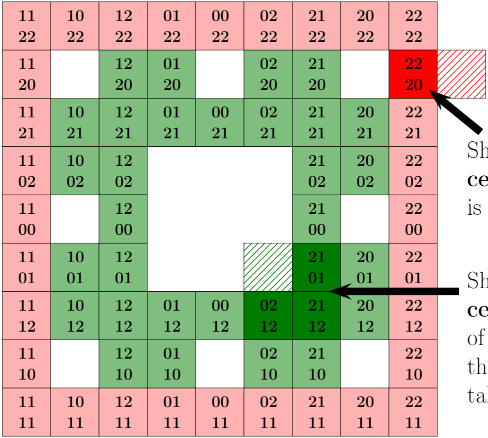

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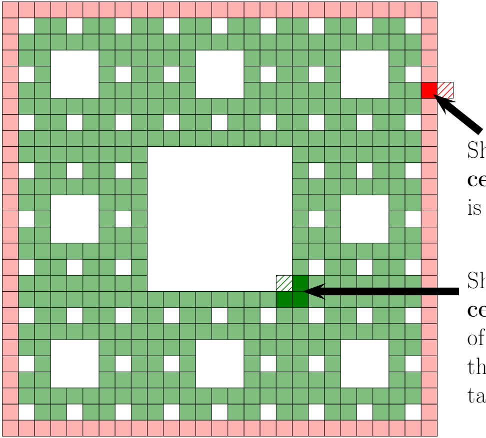

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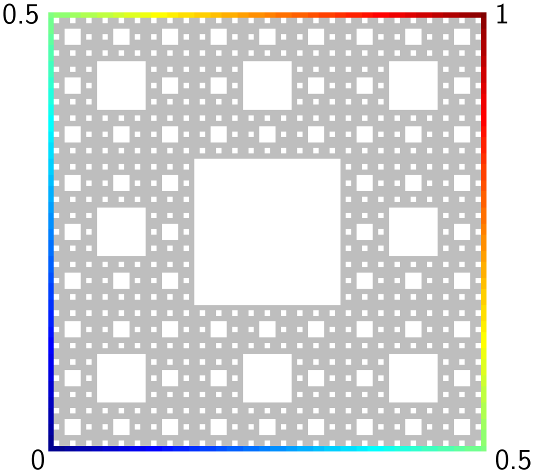
a short numerical approach

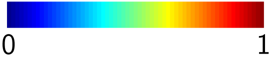
using this definition of a boundary, we get for example:



with the boundary function:

$$(x, y) \mapsto \frac{x + y}{2}$$

and values:



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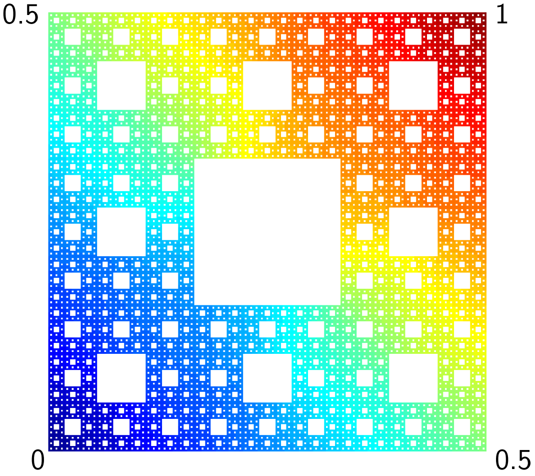

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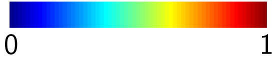

Thank you for your attention!