A Noncommutative Metric

Therese Landry
Faculty Advisor: Dr. Michel Lapidus

University of California, Riverside

tland004@ucr.edu

Cornell Fractals 6, June 15, 2017

The space of continuous complex-valued functions supported on the Cantor set, $C(\mathcal{C})$, is an Approximately Finite Algebra.

Definition

An *Approximately Finite*, or *AF*, *Algebra* is an inductive limit of finite dimensional C*-algebras.

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with the product topology.

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- η_n is a projection
- $u_n := 2\eta_n - 1_{C(\mathcal{C})}$ is a self-adjoint unitary in $C(\mathcal{C})$

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- $C(\mathcal{C})$ is the closure of $\cup_{n\in\mathbb{N}}\mathcal{A}_n$

$C(\mathcal{C})$ is a quantum compact metric space.

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A *quantum compact metric space* $(\mathcal{A}, L)$ is a pairing of a unital C*-algebra with a Lip-norm L such that

$$\{a \in \mathfrak{sa}(\mathcal{A}) : L(a) = 0\} = \mathbb{R}1_{\mathcal{A}}$$

and mk_L metrizes the weak* topology of $\mathcal{S}(\mathcal{A})$.

A faithful tracial state can be defined on $C(\mathcal{C})$.

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- with respect to addition modulo 1, the set $\mathcal{C} = \Pi_{n \in \mathbb{N}} \mathbb{Z}^2$ is a group, hence $\mathcal{C}$ compact implies there exists a unique Haar probability measure which defines by integration a faithful tracial state λ on $C(\mathcal{C})$

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then $u_n \in \mathcal{A}_n^{\perp}$ for all $n \in \mathbb{N}$. Moreover, $\left(\prod_{j \in F} u_j\right)_{F \in \mathcal{F}}$, where $\mathcal{F}$ is the set of nonempty finite subsets of $\mathbb{N}$, is an orthonormal family of $L^2(C(\mathcal{C}), \lambda)$.

Standard ultrametrics on the abelian AF algebra of $\mathbb{C}$-valued functions on the Cantor space can be recovered from a distinguished noncommuative, or quantum, metric.

Definition

A *conditional expectation* $\mathbb{E}(\cdot|\mathcal{B}) : \mathcal{A} \to \mathcal{B}$ onto $\mathcal{B}$, where $\mathcal{A}$ is a C*-algebra and $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$, is a linear positive map of norm 1 such that for all $b, c \in \mathcal{B}$ and $a \in \mathcal{A}$ we have

$$\mathbb{E}(bac|\mathcal{B}) = b\mathbb{E}(a|\mathcal{B})c.$$

Theorem (Aguilar, Latrémolière 2015)

For all $n \in \mathbb{N}$, let $\mathcal{A}_n$ be a finite dimensional C*-algebra, $\mathcal{A}_0 = \mathbb{C}$ and $\mathcal{A}$ be the AF-algebra with faithful tracial state λ defined by the closure of $\cup_{n \in \mathbb{N}} \mathcal{A}_n$.

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$$L(a) := \sup\left\{\frac{||a - \mathbb{E}_n(a)||_{\mathcal{A}}}{\beta_n} : n \in \mathbb{N}\right\},$$

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Since $u_n \in \mathcal{A}_k^{\perp}$ *in* $L^2(C(\mathcal{C}), \lambda)$*, the lemma implies* $\mathbb{E}(u_n|\mathcal{A}_k) = 0$!

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$$\mathsf{mk}_{L^{\beta}_{(\mathcal{A}_n, \alpha_n)_{n\in\mathbb{N}}, \lambda}}(x, y) = \begin{cases} 0 \quad \textit{if} \quad x = y, \\ 2\beta(\min\{n \in \mathbb{N} : x_n \neq y_n\}) \quad \textit{otherwise}. \end{cases}$$

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By construction, $\mathrm{mk}_{L^\beta_{\mathcal{T},\lambda}}$ is an ultrametric on $\mathcal{C}$.

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