An Introduction to the Theory of Complex Dimensions and Fractal Zeta Functions

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Figure 1: The rain of complex dimensions falling from the music of the angel's fractal harp (or fractal string).

Goals

- Introducing a new class of fractal zeta functions: distance and tube zeta functions associated with bounded fractal sets in Euclidean spaces of arbitrary dimensions.
- Developing a higher-dimensional theory of complex fractal dimensions valid for arbitrary compact sets (and eventually, for suitable metric measure spaces).
- Merging of aspects of complex, spectral and harmonic analysis, geometry, and number theory of fractal sets in $\mathbb{R}^N$.

Main References for this Talk:

Background Material: M. L. Lapidus, M. van Frankenhuijsen†: *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings, research monograph, second revised and enlarged edition* (*of the 2006 edition*), Springer, New York, 2013, 593 pages. ([L-vF])

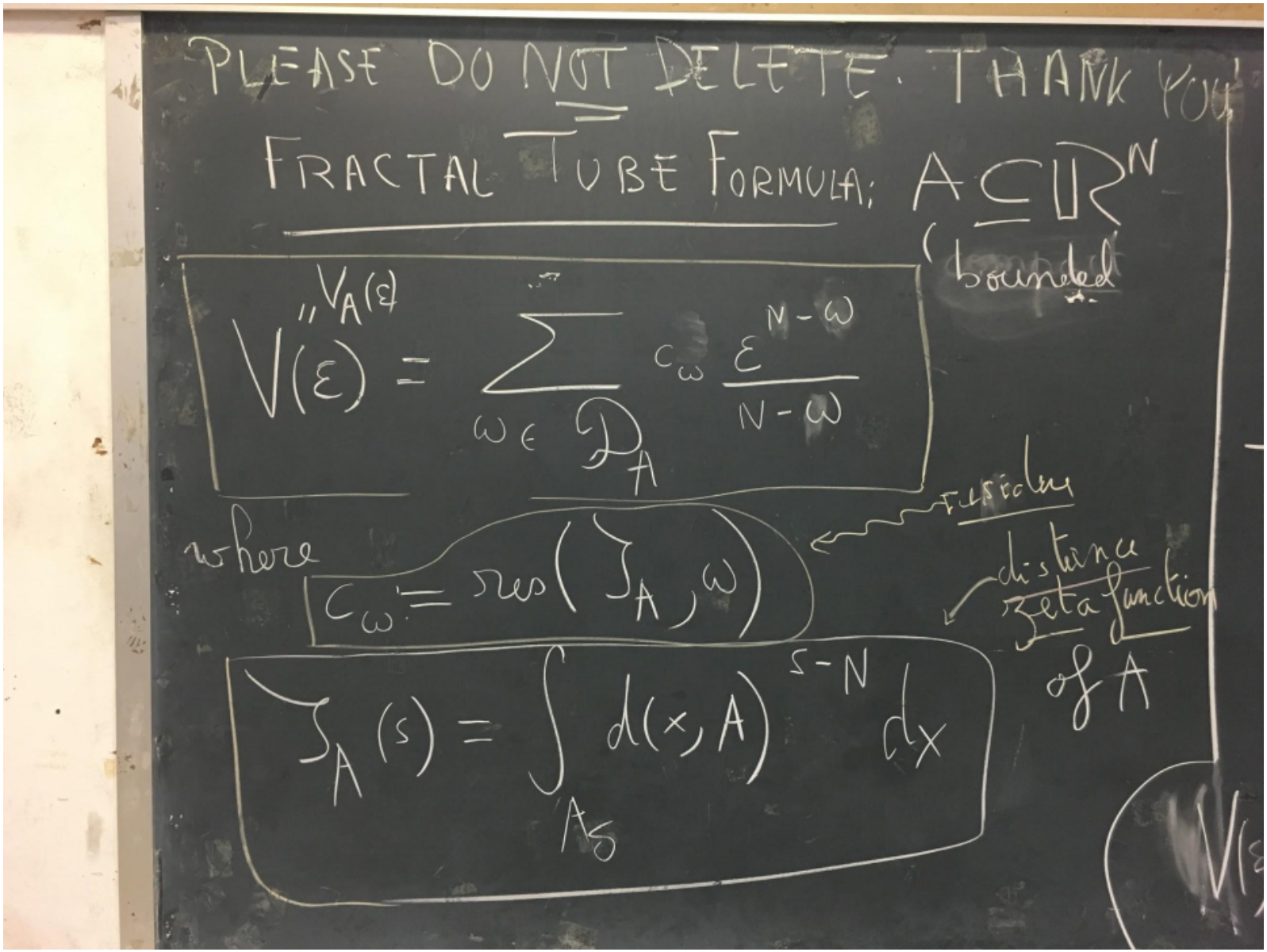
New Results: M. L. Lapidus, G. Radunović‡, D. Žubrinić‡, *Fractal Zeta Functions and Fractal Drums*: *Higher-Dimensional Theory of Complex Dimensions*, research monograph, Springer, New York, 2017, 684 pages. ([LRŽ])

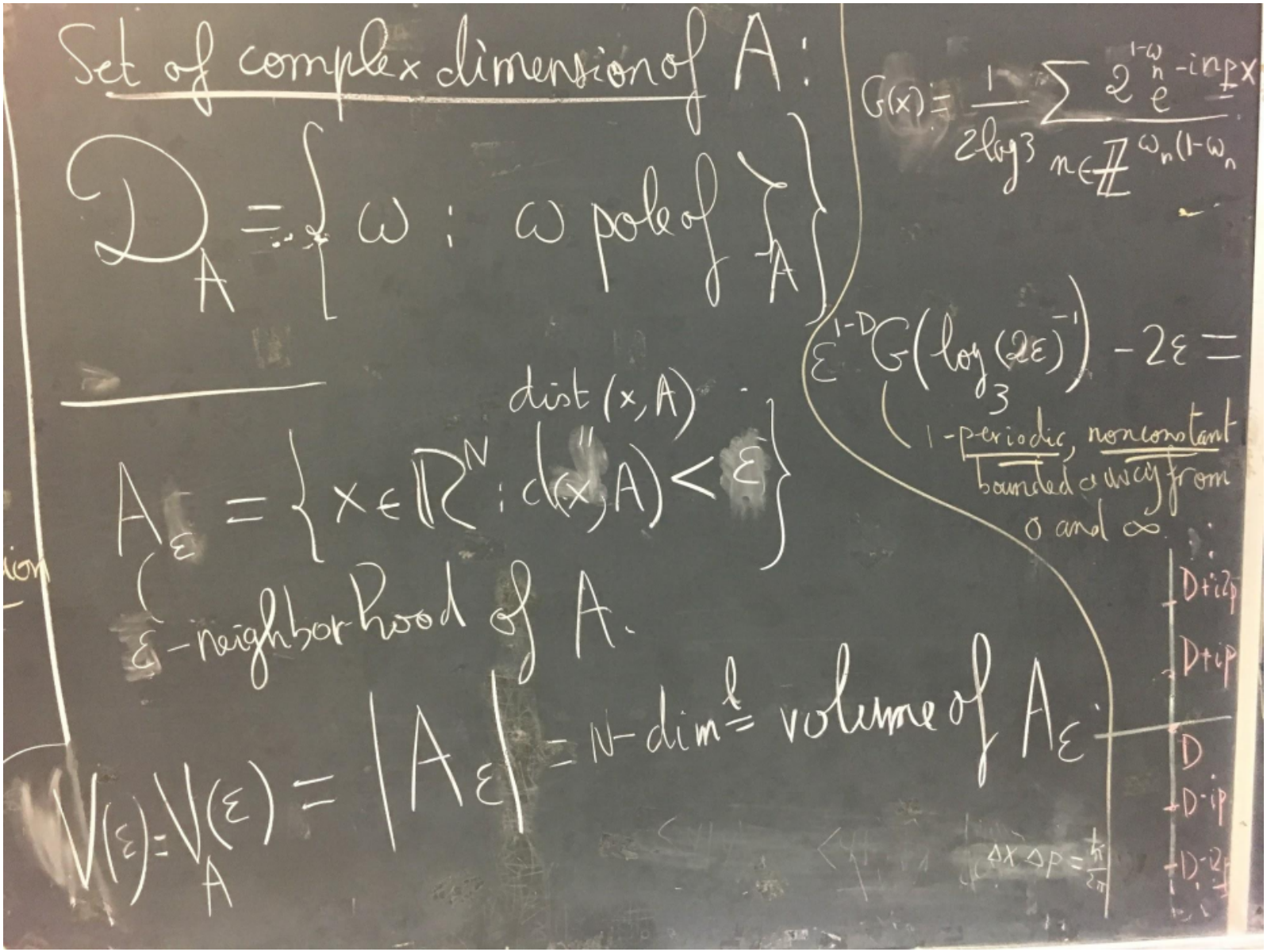
(And nine related papers by the authors of [LRŽ]; see the bibliography.)

† of Utah Valley University, USA
‡ of the University of Zagreb, Croatia

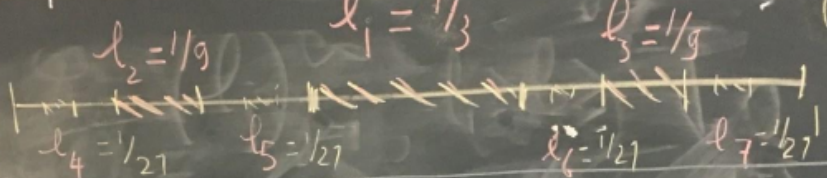
Book in Preparation:

M. L. Lapidus, *Complex Fractal Dimensions, Quantized Number Theory and Fractal Cohomology*: *A Tale of Oscillations, Unreality and Fractality*, book in preparation, approx. 350 pages, 2017.


PLEASE DO NOT DELETE. THANK YOU!
FRACTAL TUBE FORMULA: $A \subseteq \mathbb{R}^N$
(bounded)
$V_A(\varepsilon)$
$V(\varepsilon) = \sum_{\omega \in \mathcal{D}_A} c_\omega \frac{\varepsilon^{N-\omega}}{N-\omega}$
where
$c_\omega = \operatorname{res}(\zeta_A, \omega)$
residue
distance zeta function of A
$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} dx$


Set of complex dimension of A:
$\mathcal{D}_A = \{\omega : \omega \text{ pole of } \zeta_A\}$
$A_\varepsilon = \{x \in \mathbb{R}^N : d(x,A) < \varepsilon\}$
dist(x, A)
ε-neighborhood of A.
$V(\varepsilon) = V_A(\varepsilon) = |A_\varepsilon|$ = N-dim^l volume of A_ε
$G(x) = \frac{1}{2\log 3} \sum_{n\in\mathbb{Z}} \frac{2^{1-\omega_n} e^{-inpx}}{\omega_n(1-\omega_n)}$
$\varepsilon^{1-D} G(\log_3 (2\varepsilon)^{-1}) - 2\varepsilon =$
1-periodic, nonconstant
bounded away from 0 and ∞
D+i2p
D+ip
D
D-ip
D-i2p
$\Delta x \Delta p = \frac{\hbar}{2\pi}$

Expl 1: (Cantor string $\mathcal{L} = CS = (\ell_j)_{j=1}^{\infty}$)



Cantor set

$$\boxed{V_{CS}(\varepsilon) = \frac{1}{2\log 3} \sum_{\omega_n \in \mathcal{D}_{\mathcal{L}}} \frac{(2\varepsilon)^{1-\omega_n}}{\omega_n(1-\omega_n)} - 2\varepsilon}$$

where

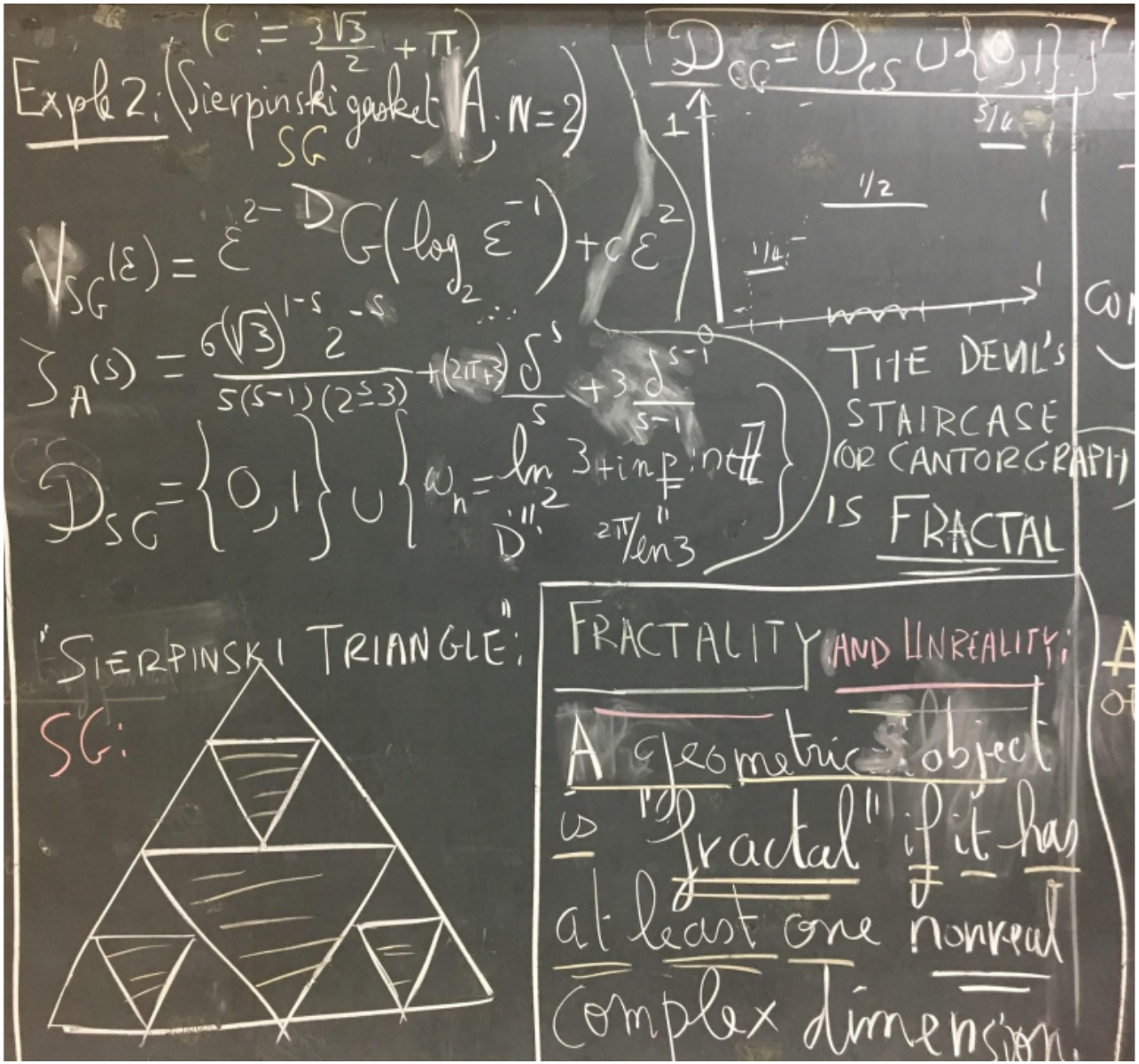
$$\mathcal{D}_{\mathcal{L}} = \{\text{poles of } \zeta_{\mathcal{L}}\} = \left\{\omega_n = \underbrace{\frac{\ln 2}{\ln 3}}_{= D} + i n \underbrace{\frac{2\pi}{\ln 3}}_{\text{oscillary period}} : n \in \mathbb{Z}\right\}$$

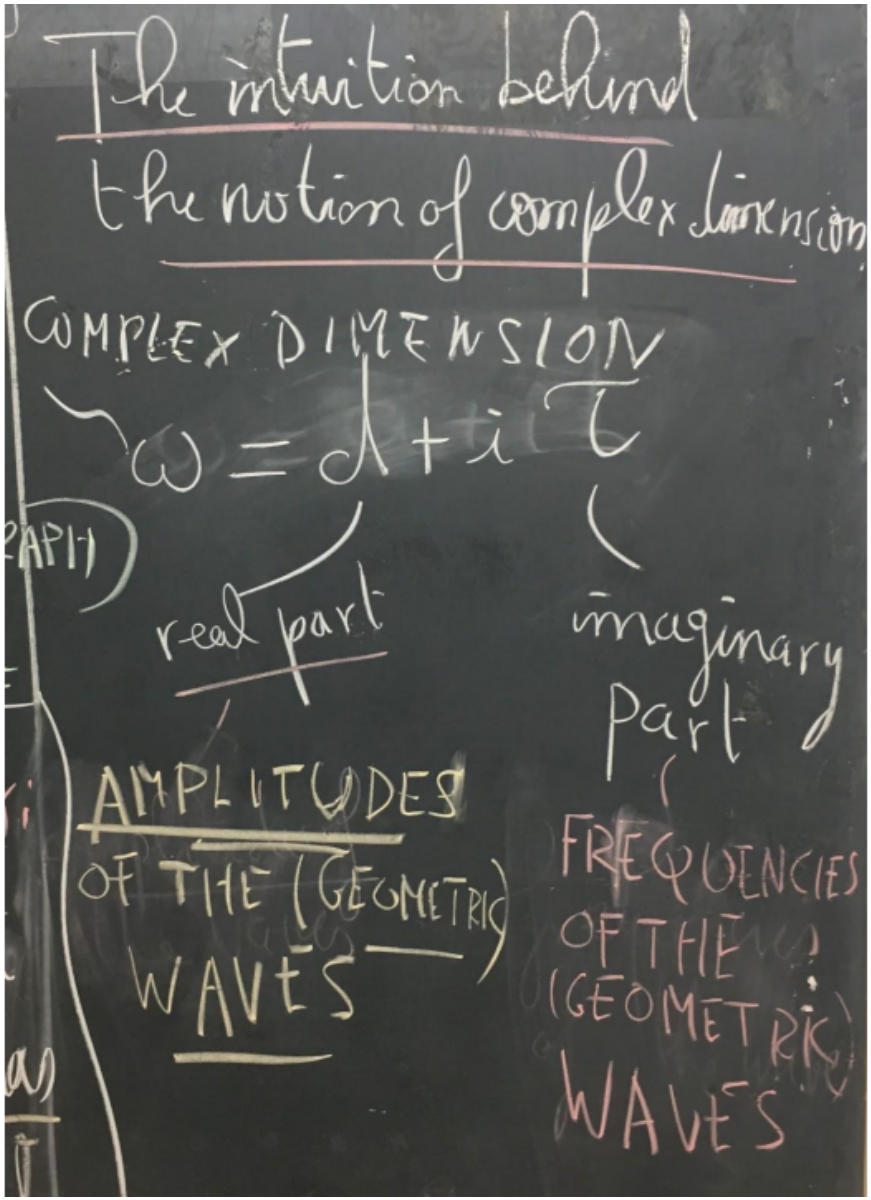
D — dimension of CS

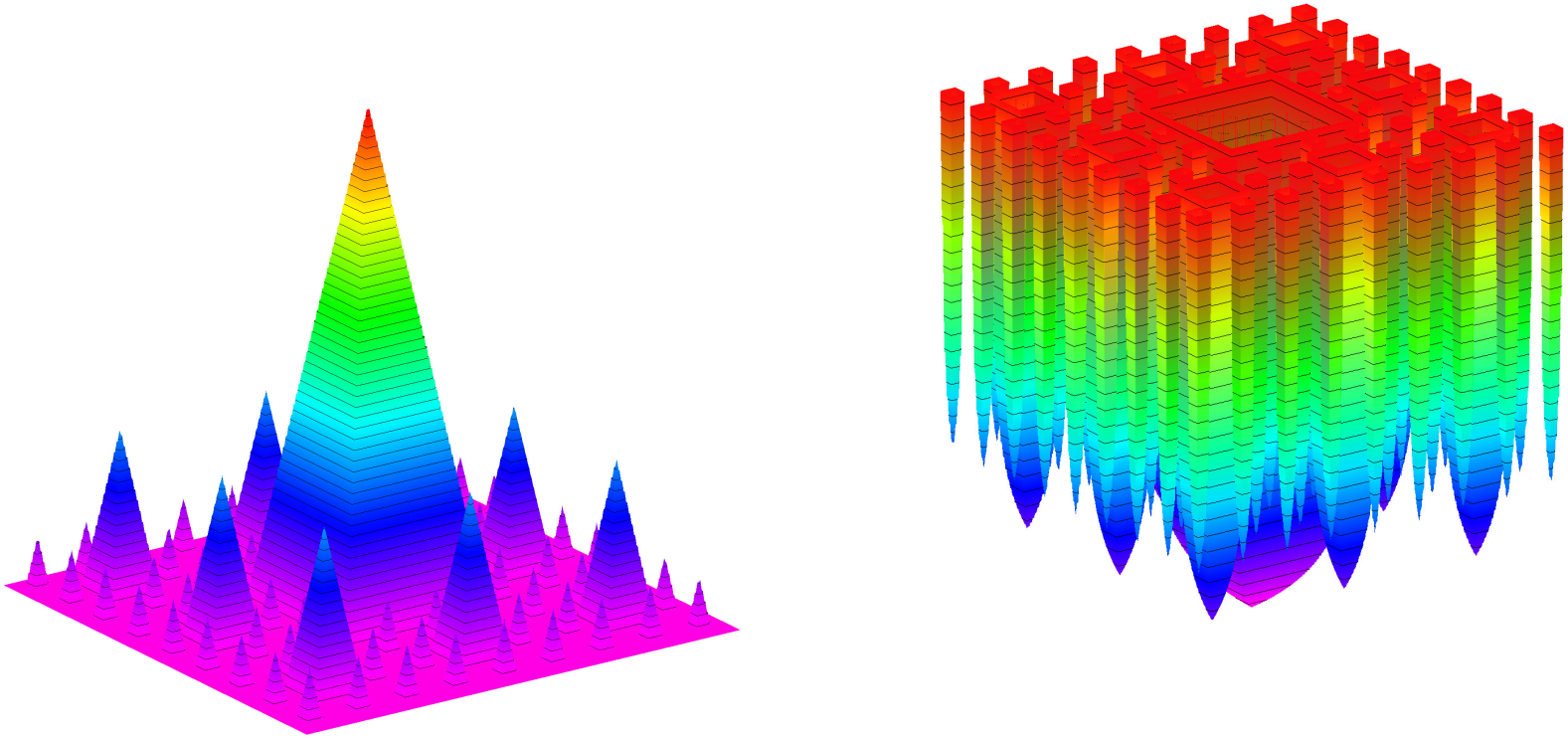
$$\boxed{\zeta_{CS}(s) = \sum_{j=1}^{\infty} \ell_j^{s} = \frac{1}{3^s - 2}}$$

geometric zeta function of $\mathcal{L}$.

$$\boxed{\zeta_A(s) = \frac{2^{1-s}}{1-s}\, \zeta_{CS}(s)}$$


Exple 2: (Sierpinski gasket A, N=2)
SG
$V_{SG}(\varepsilon) = \varepsilon^{2-D} G(\log_2 \varepsilon^{-1}) + c\varepsilon^2$
$\zeta_A(s) = \frac{6(\sqrt{3})^{1-s} 2^{-s}}{s(s-1)(2^s-3)} + (2\pi+3)\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}$
$D_{SG} = \{0,1\} \cup \{\omega_n = \ln 3 + in\frac{2\pi}{\ln 3}, n \in \mathbb{Z}\}$
$D_{CG} = D_{CS} \cup \{0,1\}$
THE DEVIL'S STAIRCASE (OR CANTOR GRAPH) IS FRACTAL
"SIERPINSKI TRIANGLE":
SG:
FRACTALITY AND UNREALITY:
A geometric object is "fractal" if it has at least one nonreal complex dimension.


The intuition behind
the notion of complex dimension
COMPLEX DIMENSION
ω = d + i τ
real part
imaginary part
AMPLITUDES
OF THE (GEOMETRIC)
WAVES
FREQUENCIES
OF THE
(GEOMETRIC)
WAVES



(a) Fractal stalagmites **(b)** Fractal stalactites

Figure 2: Stalagmites and stalactites in a fractal cave

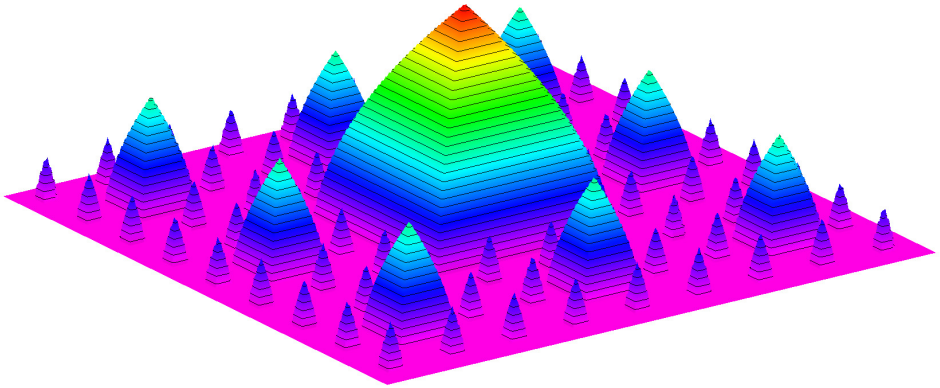


Figure 3: Other fractal stalagmites

Minkowski Content

Let $A \subset \mathbb{R}^N$ be a nonempty bounded set.

- ε-neighborhood of A,

$$A_\varepsilon = \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}.$$

- *Lower s-dimensional Minkowski content* of A, $s \geq 0$:

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0^+} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

  where $|A_\varepsilon|$ is the N-dimensional Lebesgue measure of A_ε.

- *Upper s-dimensional Minkowski content* of A:

$$\mathcal{M}^{*s}(A) := \limsup_{\varepsilon \to 0^+} \frac{|A_\varepsilon|}{\varepsilon^{N-s}}.$$

Box Dimensions

- Lower box dimension of A: $\underline{\dim}_B A = \inf\{s \geq 0 : \mathcal{M}_*^s(A) = 0\}$.
- Upper box dimension of A:

$$\overline{\dim}_B A = \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}.$$

- $\dim_H A \leq \underline{\dim}_B A \leq \overline{\dim}_B A \leq N$,
  where $\dim_H A$ denotes the Hausdorff dimension of A.
- If $\underline{\dim}_B A = \overline{\dim}_B A$, we write $\dim_B A$, the box dimension of A.
- If there is $d \geq 0$ such that

$$0 < \mathcal{M}_*^d(A) \leq \mathcal{M}^{*d}(A) < \infty,$$

  we say A is Minkowski nondegenerate. Clearly, $d = \dim_B A$.
- If $|A_\varepsilon| \asymp \varepsilon^\sigma$ for all sufficiently small ε and some $\sigma \leq N$, then A is Minkowski nondegenerate and

$$\dim_B A = N - \sigma.$$

Minkowski Measurable and Nondegenerate Sets

If $\mathcal{M}_*^s(A) = \mathcal{M}^{*s}(A)$ for some s, we denote this common value by $\mathcal{M}^s(A)$ and call it the *s-dimensional Minkowski content* of A.

Furthermore, if $\mathcal{M}^d(A) \in (0, \infty)$ for some $d \geq 0$, then A is said to be Minkowski measurable, with Minkowski content $\mathcal{M}^d(A)$ (often simply denoted by $\mathcal{M}$). Clearly, we then have $d = \dim_B A$.

Example (The Cantor set)

The triadic Cantor set A has box dimension $\dim_B A = \log 2/\log 3$. A is *not* Minkowski measurable (Lapidus & Pomerance, 1993).

Example (a-set)

Let $A = \{k^{-a} : k \in \mathbb{N}\}$ be the a-set, for $a > 0$. Then $\dim_B A = 1/(1+a)$ and A is Minkowski measurable (L., 1991).

The Triadic Cantor Set

$\frac{|A_\varepsilon|}{\varepsilon^{1-d}} = G(\log_3 \varepsilon^{-1}) + O(\varepsilon^d)$ as $\varepsilon \to 0^+$. Here, $d = \log_3 2$.

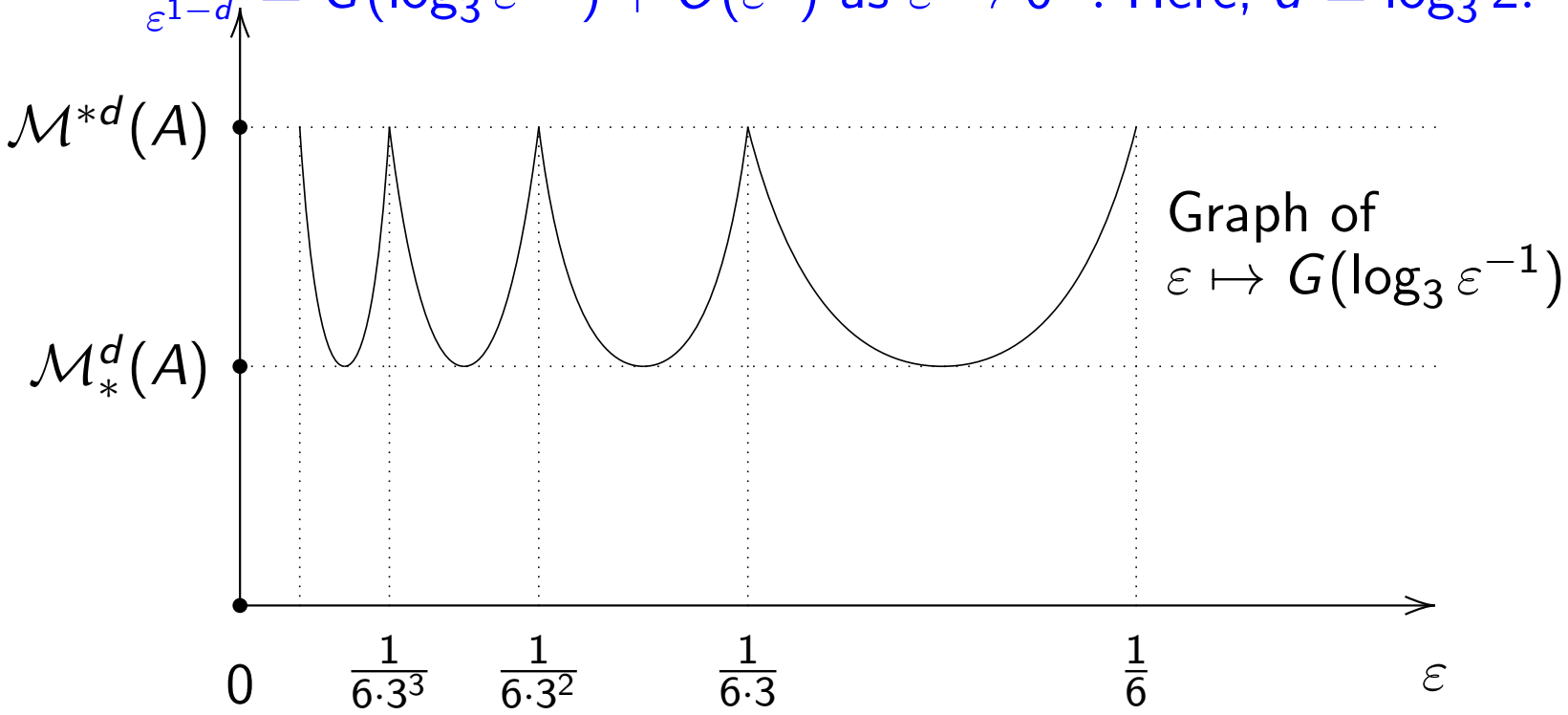


Figure 4: The oscillating nature of the function $\varepsilon \mapsto |A_\varepsilon|/\varepsilon^{1-d}$ near $\varepsilon = 0$ for the triadic Cantor set A, with $d := \dim_B A = \log_3 2$. Then, A *is Minkowski nondegenerate, but is not Minkowski measurable* [Lapidus & Pomerance, 1993]. The function $G(\tau)$ is $\log 3$-periodic. (See [Lapidus & van Frankenhuijsen, 2000, 2006 & 2013] for much more detailed information.)

Singular Function Generated by the Cantor Set

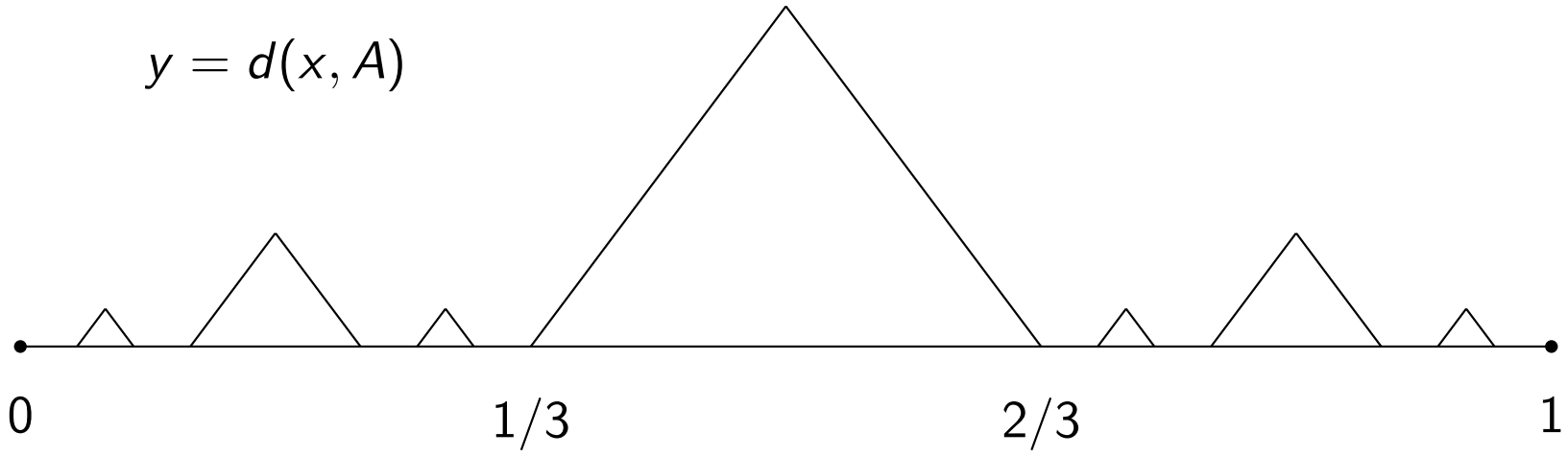


Figure 5: The graph of the distance function $x \mapsto d(x, A)$, where A is the classic ternary (or triadic) Cantor set $C^{(1/3)}$.

Singular Function Generated by the Cantor Set

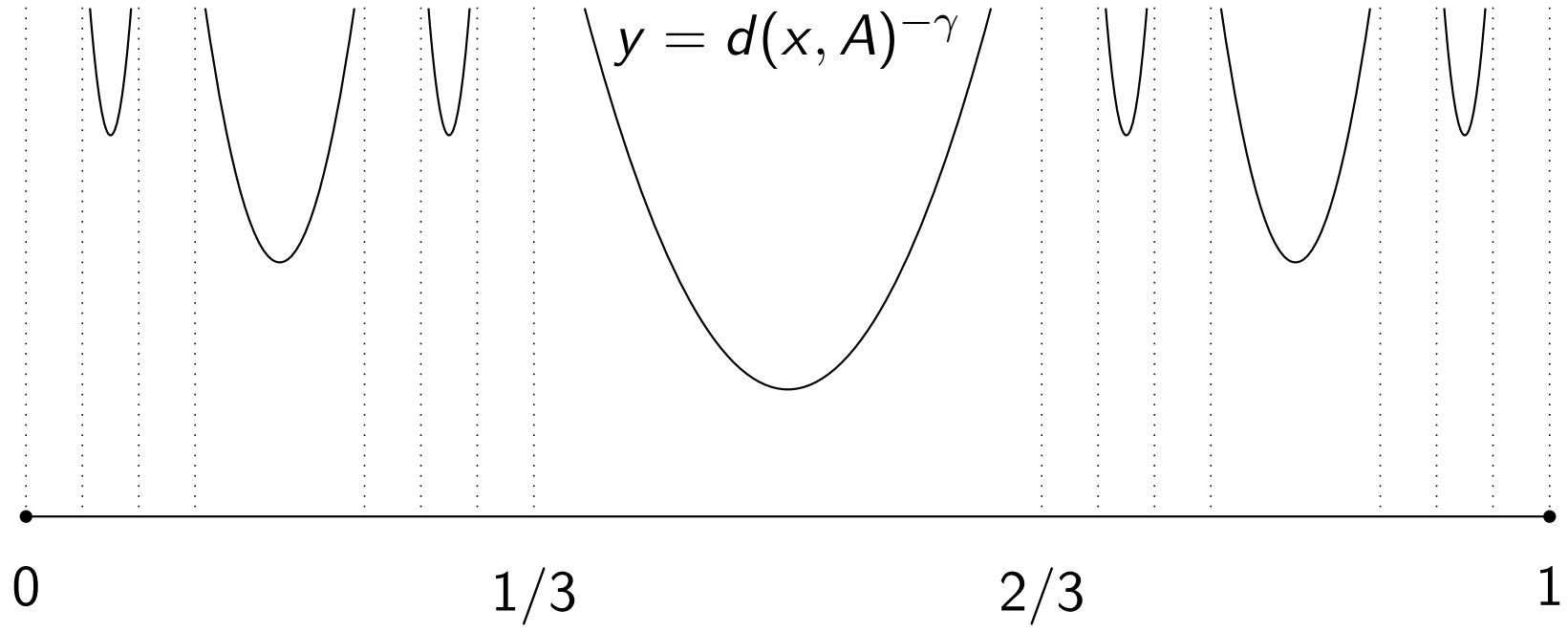
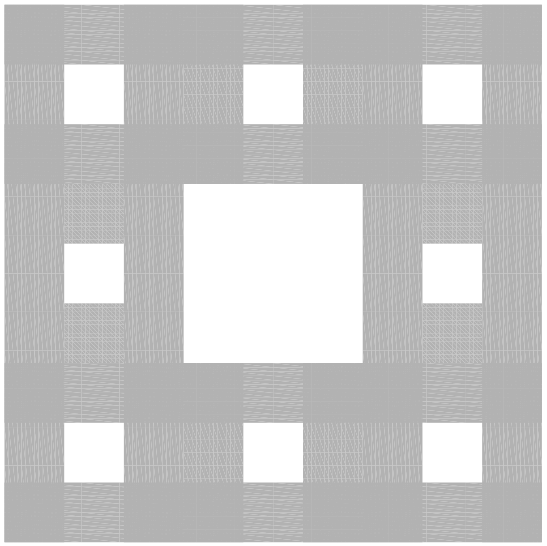


Figure 6: For the triadic Cantor set A, *the function* $y = d(x, A)^{-\gamma}$, $x \in (0, 1)$, *is Lebesgue integrable if and only if* $\gamma < 1 - \log 2/\log 3$. Here, $\dim_H A = \dim_B A = \log 2/\log 3$, where $\dim_H A$ denotes the Hausdorff dimension of A.



The Sierpinski carpet A (two iterations are shown);

$$\dim_H A = \dim_B A = \frac{\log 8}{\log 3},$$

A is Minkowski nondegenerate, but not Minkowski measurable (L., 1993).

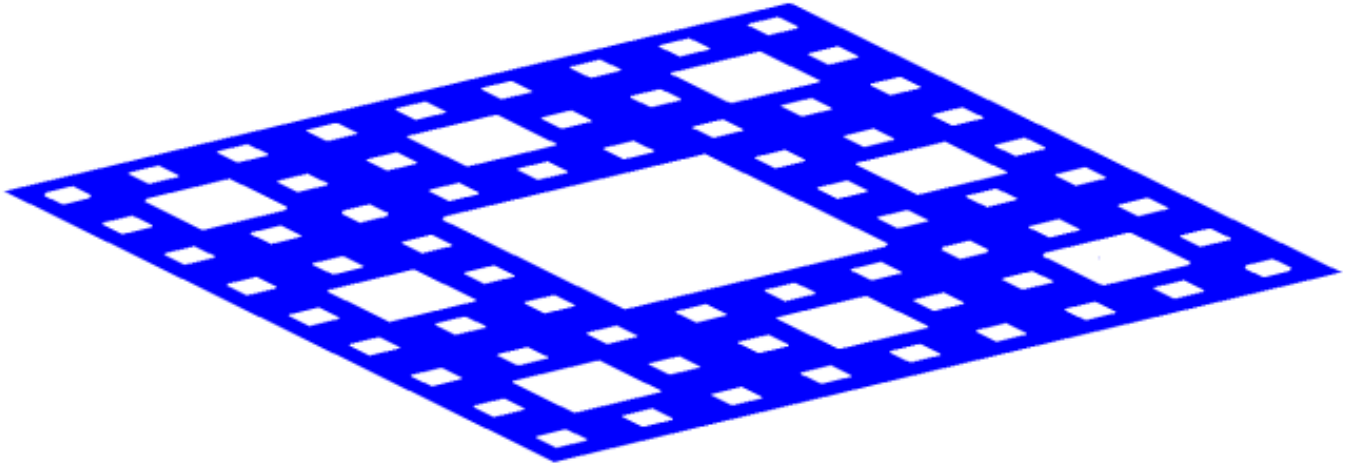
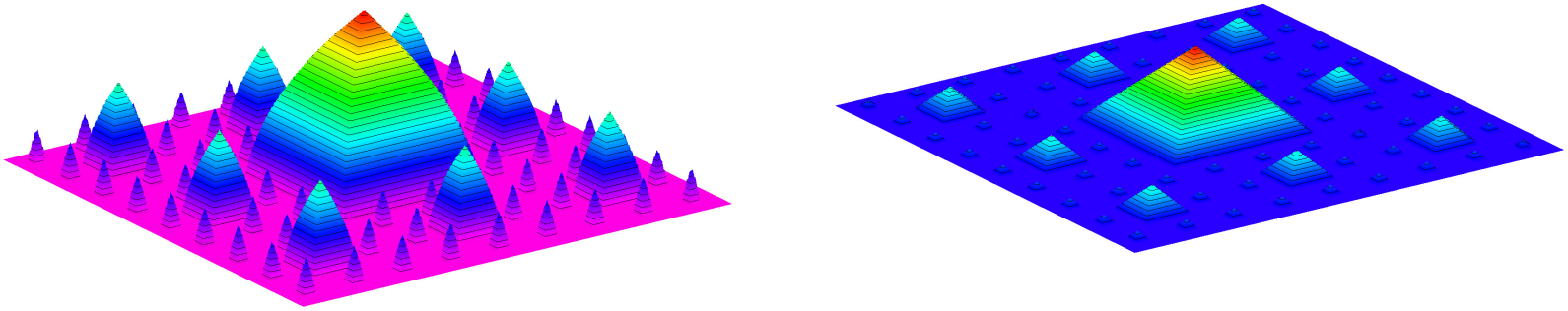


Figure 7: The classic self-similar Sierpinski carpet



(a) The Hölder case **(b)** The Lipschitz case

Figure 8: The Sierpinski stalagmites The graph of $f(x) = d(x, A)^r$, where $r \in (0, 1)$ or $r \geq 1$, respectively. Here, $r = 0.5$ **(a)** or $r = 1.3$ **(b)**.

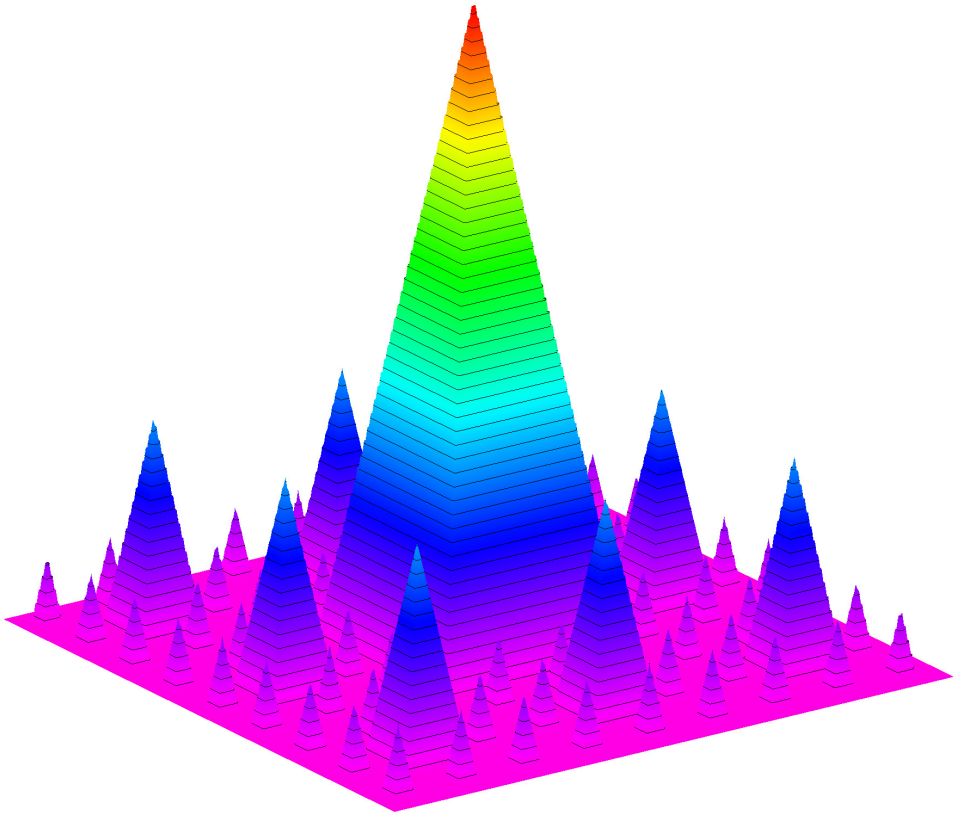


Figure 9: Fractal stalagmites associated with the Sierpinski carpet.

The figure on the previous slide depicts the graph of the distance function $y = d(x, A)$, defined on the unit square, where A is the Sierpinski carpet. Only the first three generations of the countable family of pyramidal tents (called *stalagmites*) are shown.

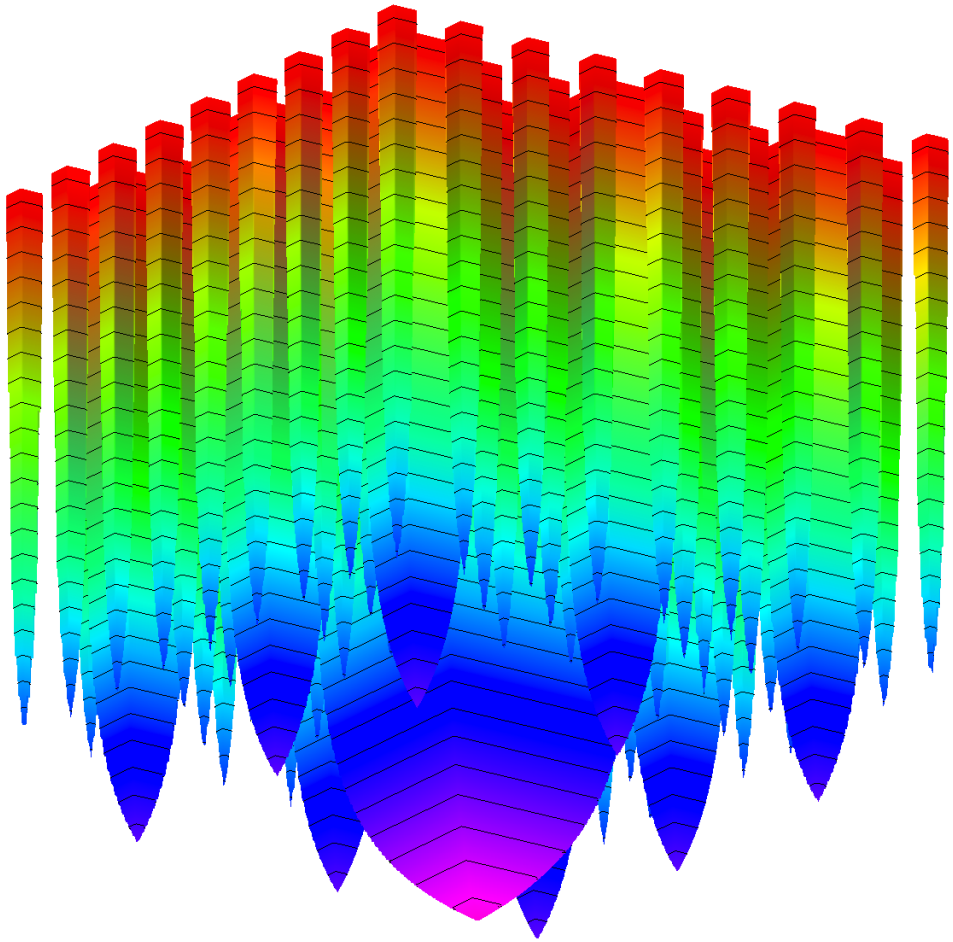


Figure 10: Fractal stalactites associated with the Sierpinski carpet.

The figure on the previous slide shows the graph of the function $y = d(x, A)^{-\gamma}$, defined on the unit square, where A is the Sierpinski carpet. Since A is known to be Minkowski nondegenerate, this function is Lebesgue integrable if and only if $\gamma \in (-\infty, 2 - D), D = \dim_B A = \log_3 8$. For $\gamma > 0$, the graph consists of countably many connected components, called *stalactites*, all of which are unbounded.

Definition of Fractal Strings

- $\mathcal{L} = (\ell_j)_{j\geq 1}$ a fractal string (L., 1991, L. & Pomerance, 1993): a nonincreasing sequence of positive numbers (ℓ_j) such that $\sum_j \ell_j < \infty$. [Alternatively, $\mathcal{L}$ can be viewed as a sequence of *scales* or as the *lengths* of the connected components (open intervals) of a *bounded open set* $\Omega \subset \mathbb{R}$.]

- The zeta function (*geometric zeta function*) of the fractal string $\mathcal{L}$ is the Dirichlet series:

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} (\ell_j)^s,$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s$ large enough.

- Consider the open intervals $I_j = (a_j, a_{j-1})$ for $j \geq 1$, where

$$a_j := \sum_{k>j} \ell_k, \quad \text{and } \ell_j := |I_j|.$$

- Define $A = \{a_j\}$. Then A is a bounded set, $A \subset \mathbb{R}$, and $a_j \to 0$ as $j \to \infty$. The set $A = A_{\mathcal{L}}$ is introduced in [LRŽ].

- $\overline{\dim}_B A$ is defined via the upper Minkowski content of A, as usual.

Theorem (L)

The abscissa of convergence of $\zeta_{\mathcal{L}}$ is equal to $\overline{\dim}_B A$:

$$\overline{\dim}_B A = \inf\left\{\alpha > 0 : \sum_{j=1}^{\infty} (\ell_j)^{\alpha} < \infty\right\}.$$

Theorem (L, L-vF)

- *$\zeta_{\mathcal{L}}(s)$ is holomorphic on the right half-plane $\{\operatorname{Re} s > \overline{\dim}_B A\}$;*
- *The lower bound $\overline{\dim}_B A$ is optimal, both from the point of view of the absolute convergence and the holomorphic continuation.*
- *Moreover, if $s \in \mathbb{R}$ and $s \to \overline{\dim}_B A$ from the right, then $\zeta_{\mathcal{L}}(s) \to +\infty$.*

Corollary

The abscissa of holomorphic continuation and the abscissa of (absolute) convergence of $\zeta_{\mathcal{L}}$ both coincide with the (upper) Minkowski dimension of $\mathcal{L}$:

$$D_{hol}(\zeta_{\mathcal{L}}) = D(\zeta_{\mathcal{L}}) = \overline{\dim}_B A.$$

Remarks: In the above discussion, A could be replaced by $\partial\Omega$, the boundary of any geometric realization of $\mathcal{L}$ by a bounded open set $\Omega \subset \mathbb{R}$.

Definition of the Distance Zeta Function

Let $A \subset \mathbb{R}^N$ be an arbitrary bounded set, and let $\delta > 0$ be fixed. As before, A_δ denotes the δ-neighborhood of A.

Definition (L., 2009; LRŽ, 2013)

The distance zeta function of A is defined by

$$\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x,$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

Remarks:

- *For $s \in \mathbb{C}$ such that $\operatorname{Re} s < N$, the function $d(x, A)^{s-N}$ is singular on A.*

- The inequality $\delta < \delta_1$ implies that

$$\zeta_A(s; A_{\delta_1}) - \zeta_A(s) = \int_{A_{\delta,\delta_1}} d(x, A)^{s-N} \mathrm{d}x$$

is an entire function. *As a result, the definition of $\zeta_A = \zeta_A(\cdot, A_\delta)$ does not depend on δ in an essential way.* In particular, the complex dimensions of A (i.e., the poles of ζ_A) do not depend on δ.

Zeta Function of the Set A Associated to a Fractal String $\mathcal{L}$

- Let $\mathcal{L} = (\ell_j)$ be a fractal string, and $A = (a_j)$, $a_j = \sum_{k \geq j} \ell_k$.
- We would like to compare $\zeta_{\mathcal{L}}(s) = \sum_j (\ell_j)^s$ and

$$\zeta_A(s) = \int_{-\delta}^{1+\delta} d(x, A)^{s-1} \mathrm{d}x, \text{ for } \delta \geq \ell_1/2.$$

- The zeta functions of $\mathcal{L}$ and A are 'equivalent', $\zeta_A(s) \sim \zeta_{\mathcal{L}}(s)$, in the following sense:

$$\zeta_A(s) = a(s)\zeta_{\mathcal{L}}(s) + b(s),$$

where $a(s)$ and $b(s)$ are explicit functions which are holomorphic on $\{\operatorname{Re} s > 0\}$, and $a(s) \neq 0$ for all such s. It follows that (when they exist) *the meromorphic extensions of* $\zeta_A(s)$ *and* $\zeta_{\mathcal{L}}(s)$ *have the same sets of poles in* $\{\operatorname{Re} s > 0\}$ (i.e., *the same set of visible complex dimensions* up to 0).

Definition

The complex dimensions of a fractal string $\mathcal{L}$ (L & vF, 1996) are defined as the poles of $\zeta_{\mathcal{L}}$.

Definition

The complex dimensions of a bounded set $A \subset \mathbb{R}^N$ are defined as the poles of ζ_A (L., 2009; LRŽ, 2013).

Remark: We assume here that the zeta functions involved have a meromorphic extension (necessarily unique, by the principle of analytic continuation) to some suitable region $U \subset \mathbb{C}$.

Visible complex dimensions: the poles of ζ_A in U.

Holomorphy Half-Plane of the Distance Zeta Function

Let A be a nonempty bounded set in $\mathbb{R}^N$; given δ a fixed positive number, let $\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} \mathrm{d}x$ as before.

Theorem (LRŽ)

- *$\zeta_A(s)$ is holomorphic on the right half-plane $\{\operatorname{Re} s > \overline{\dim}_B A\}$; the lower bound $\overline{\dim}_B A$ is optimal from the point of view of the convergence of the Lebesgue integral defining ζ_A.*
- *Moreover, if $D = \dim_B A$ exists, $D < N$, and $\mathcal{M}_*^D(A) > 0$, then $\zeta_A(s) \to +\infty$ as $s \in \mathbb{R}$ and $s \to D^+$; so that the lower bound $\overline{\dim}_B A$ is also optimal from the point of view of the holomorphic continuation.*

Remark: If $s \in \mathbb{R}$ and $s < \overline{\dim}_B A$, then $\zeta_A(s) = +\infty$.

Corollary (LRŽ)

The abscissa of (absolute) convergence of ζ_A is equal to $\overline{\dim}_B A$, the (upper) Minkowski dimension of A:

$$D(\zeta_A) := \inf\left\{\alpha \in \mathbb{R} : \int_{A_\delta} d(x,A)^{\alpha-N} dx < \infty\right\}$$
$$= \overline{\dim}_B A.$$

Corollary (LRŽ)

Assume that $D = \dim_B A$ exists, $D < N$, and $\mathcal{M}_*^D(A) > 0$. Then the abscissa of holomorphic continuation and the abscissa of (absolute) convergence of ζ_A both coincide with the (upper) Minkowski dimension of A :

$$D_{hol}(\zeta_A) = D(\zeta_A) = \overline{\dim}_B A.$$

Definition

The abscissa of holomorphic continuation of ζ_A is given by

$$D_{hol}(\zeta_A) := \inf\{\alpha \in \mathbb{R} : \zeta_A(s) \text{ is holomorphic on } \operatorname{Re} s > \alpha\}$$

Furthermore, the open half-plane $\{\operatorname{Re} s > D_{hol}(\zeta_A)\}$ is called the holomorphy half-plane of ζ_A, while $\{\operatorname{Re} s > D(\zeta_A)\}$ is called the half-plane of (absolute) convergence of ζ_A.

Remark: In general, we have

$$D_{hol}(\zeta_A) \leq D(\zeta_A) = \overline{\dim}_B A.$$

Residue of the Distance Zeta Function at $D = \dim_B A$

We assume that $\zeta_A(s) = \zeta_A(s, A_\delta)$ can be meromorphically extended to a neighborhood of $D := \dim_B A$, and $D < N$. *We write ζ_A or $\zeta_A(\cdot, A_\delta)$, interchangeably*.

Theorem (LRŽ)

If $\mathcal{M}^{*D}(A) < \infty$, then $s = D$ is a simple pole of ζ_A and

$$(N - D)\mathcal{M}_*^D(A) \leq \operatorname{res}(\zeta_A(\cdot, A_\delta), D) \leq (N - D)\mathcal{M}^{*D}(A).$$

The value of $\operatorname{res}(\zeta_A(\cdot, A_\delta), D)$ does not depend on $\delta > 0$.

Remark: For the triadic Cantor set, we have strict inequalities.

Theorem (LRŽ)

If A is Minkowski measurable (i.e., $\mathcal{M}^D(A)$ exists and $\mathcal{M}^D(A) \in (0, \infty)$), then

$$\operatorname{res}(\zeta_A(\cdot, A_\delta), D) = (N - D)\mathcal{M}^D(A).$$

Tube Zeta Function of a Fractal Set

Let $A \subset \mathbb{R}^N$ be an arbitrary bounded set, and let $\delta > 0$ be fixed.

Definition

The tube zeta function of A associated with the tube function $t \mapsto |A_t|$, is given by (for some fixed, small $\delta > 0$)

$$\tilde{\zeta}_A(s) = \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t,$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

Remark: The choice of δ is unimportant, from the point of view of the theory of complex dimensions. Indeed, changing δ amounts to adding an entire function to $\tilde{\zeta}_A$.

The next result follows from its counterpart stated earlier for the distance zeta function ζ_A (see the sketch of the proof given below):

Corollary (LRŽ)

If $D = \dim_B A$ exists, $D < N$ and $\tilde{\zeta}_A$ has a meromorphic extension to a neighborhood of $s = D$, then

$$\mathcal{M}_*^D(A) \leq \operatorname{res}(\tilde{\zeta}_A, D) \leq \mathcal{M}^{*D}(A).$$

In particular, if A is Minkowski measurable, then

$$\operatorname{res}(\tilde{\zeta}_A, D) = \mathcal{M}^D(A).$$

The proof of the previous corollary rests on the following identity, which is valid on $\{\operatorname{Re} s > \overline{D}\}$, where $\overline{D} = \overline{\dim}_B A$:

$$\zeta_A(s, A_\delta) = \delta^{s-N}|A_\delta| + (N - s)\tilde{\zeta}_A(s).$$

Remark: It follows from the above equation that if $D < N$, then $\tilde{\zeta}_A$ has a meromorphic extension to a given domain $U \subset \mathbb{C}$ iff ζ_A does. In particular, $\tilde{\zeta}_A$ and ζ_A *have the same* (*visible*) *complex dimensions*; that is, the same poles within U.

Residues of Fractal Zeta Functions of Generalized Cantor Sets

Example (1)

For the generalized Cantor sets $A = C^{(a)}$, $a \in (0, 1/2)$, we have $D(a) = \dim_B C^{(a)} = \log_{1/a} 2$. Moreover,

$$\begin{aligned} \mathcal{M}_*^D(A) &= \frac{1}{D}\left(\frac{2D}{1-D}\right)^{1-D}, \\ \mathcal{M}^{*D}(A) &= 2(1-a)\left(\frac{1}{2}-a\right)^{D-1}, \end{aligned}$$

and

$$\operatorname{res}(\tilde{\zeta}_A, D) = \frac{2}{\log 2}\left(\frac{1}{2}-a\right)^{D}.$$

Example (1 continued)

For all $a \in (0, 1/2)$, we have

$$\mathcal{M}_*^D(A) < \operatorname{res}(\tilde{\zeta}_A(s), D) < \mathcal{M}^{*D}(A).$$

Also,

$$\operatorname{res}(\zeta_A, D) = (1 - D) \operatorname{res}(\tilde{\zeta}_A, D).$$

Remark: With this notation, the classic ternary Cantor set is just $C^{(1/3)}$. For any $a \in (0, 1/2)$, the generalized Cantor set $C^{(a)}$ is constructed in much the same way as $C^{(1/3)}$, by removing open "middle a-intervals" at each stage of the construction.

Residues and Minkowski Contents for a-strings, $a > 0$

Example (2)

The a-string associated with $A := \{k^{-a} : k \in \mathbb{N}\}$, $a > 0$, is given by

$$\mathcal{L} = (\ell_j)_{j \geq 1}, \quad \ell_j = j^{-a} - (j+1)^{-a}.$$

We have:

$$D(a) = \dim_B A = \frac{1}{1+a},$$

$$\operatorname{res}(\tilde{\zeta}_A, D) = \mathcal{M}^D(A) = \frac{2^{1-D}}{D(1-D)} a^D,$$

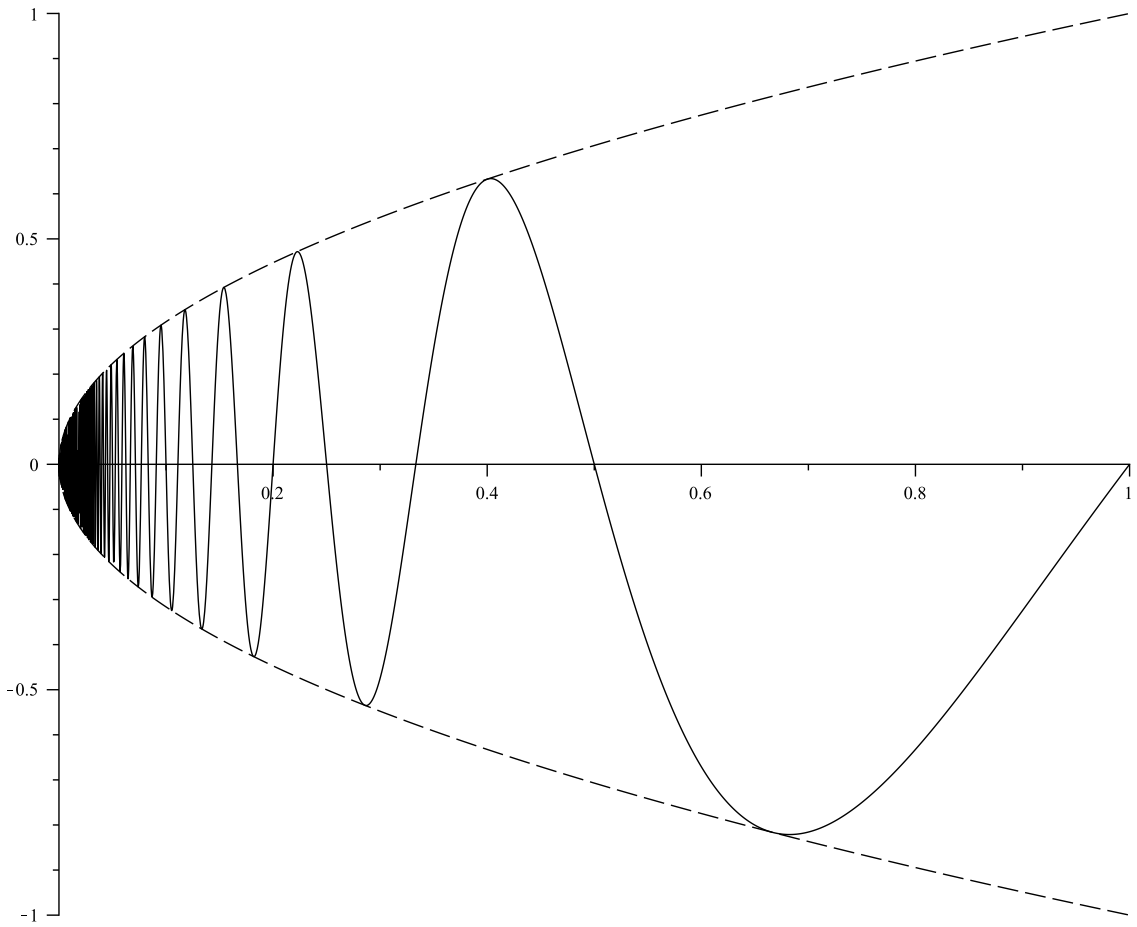
$$\operatorname{res}(\zeta_A, D) = (1-D)\mathcal{M}^D(A) = \frac{2^{1-D} a^D}{D},$$

and

$$\operatorname{res}(\zeta_{\mathcal{L}}, D) = a^D.$$

Definition (1)

Let $\alpha > 0$ and $\beta > 0$. The standard (α, β)-chirp is the graph of $y = x^{\alpha} \sin x^{-\beta}$ near the origin (here $\alpha = 1/2$, $\beta = 1$):

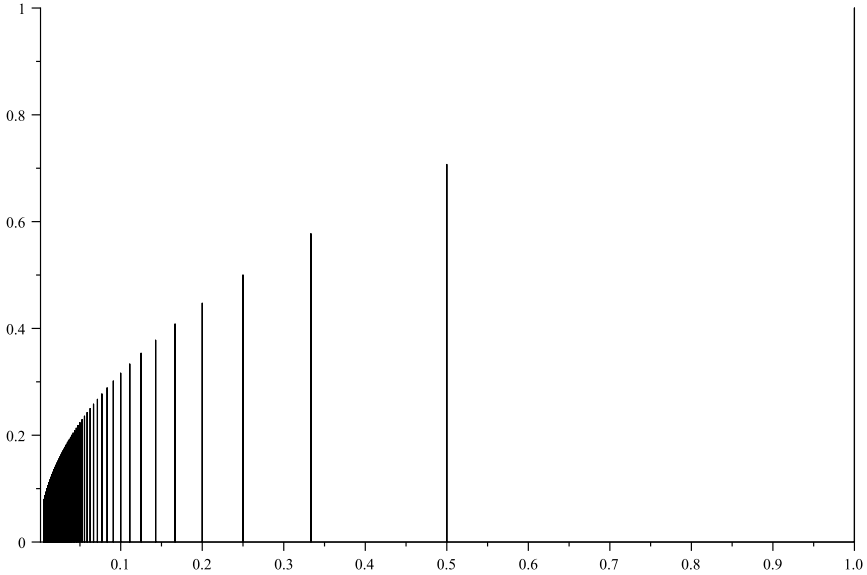


Definition of the (α, β)-chirp

Definition

Let $\alpha > 0$ and $\beta > 0$. The geometric (α, β)-chirp is the following countable union of vertical intervals in the plane ('approximation' of the standard (α, β)-chirp):

$$\Gamma(\alpha, \beta) = \bigcup_{k \in \mathbb{N}} \{k^{-1/\beta}\} \times (0, k^{-\alpha/\beta}).$$



Distance Zeta Function of Geometric Chirps

The distance zeta function of $\Gamma(\alpha, \beta)$ can be computed as follows:

$$\zeta_{\Gamma(\alpha,\beta)}(s) \sim \frac{1}{s-1} \sum_{k=1}^{\infty} k^{-\frac{\alpha}{\beta}-(1+\frac{1}{\beta})(s-1)},$$

where, as before, we define

$$\zeta_A(s) \sim f(s) \Leftrightarrow f(s) = a(s)\zeta_A(s) + b(s),$$

with $a(s)$, $b(s)$ holomorphic on $\{\operatorname{Re} s > r\}$, for some $r < \overline{\dim}_B A$, and $a(s) \neq 0$ for all such s.

The series converges iff $\operatorname{Re} s > \max\{1, 2 - \frac{1+\alpha}{1+\beta}\}$; hence,

$$\overline{\dim}_B \Gamma(\alpha, \beta) = \max\{1, 2 - \frac{1+\alpha}{1+\beta}\}.$$

This is the analog of Tricot's formula (which was originally proved for the standard (α, β)-chirp).

Minkowski Measurable Sets

Theorem (LRŽ (Minkowski measurable case))

Given $A \subset \mathbb{R}^N$, assume that there exist $\alpha > 0$, $\mathcal{M} \in (0, \infty)$ and $D \geq 0$ such that the tube function $t \mapsto |A_t|$ satisfies

$$|A_t| = t^{N-D}\left(\mathcal{M} + O(t^{\alpha})\right) \quad \text{as } t \to 0^+.$$

Then A is Minkowski measurable, and we have:

$$\dim_B A = D, \ \mathcal{M}^D(A) = \mathcal{M}, \ \text{and } D(\tilde{\zeta}_A) = D.$$

Furthermore, $\tilde{\zeta}_A$ has a (unique) meromorphic extension to (at least) $\{\operatorname{Re} s > D - \alpha\}$.

Moreover, the pole $s = D$ is unique, simple, and $\operatorname{res}(\tilde{\zeta}_A, D) = \mathcal{M}$.

Remark: Provided $D < N$, the exact same results hold for ζ_A, the distance zeta function of A. Then, we have instead

$$\operatorname{res}(\zeta_A, D) = (N - D)\mathcal{M}.$$

Minkowski Nonmeasurable Sets

Theorem (LRŽ (Minkowski nonmeasurable case))

Given $A \subset \mathbb{R}^N$, assume that there exist $D \geq 0$, a nonconstant periodic function $G : \mathbb{R} \to \mathbb{R}$ with minimal period $T > 0$, and $\alpha > 0$, such that

$$|A_t| = t^{N-D}\left(G(\log t^{-1}) + O(t^{\alpha})\right) \quad \text{as } t \to 0^+.$$

Then we have:

$$\dim_B A = D, \mathcal{M}_*^D(A) = \min G, \mathcal{M}^{*D}(A) = \max G, \text{and } D(\tilde{\zeta}_A) = D.$$

Furthermore, $\tilde{\zeta}_A(s)$ has a (unique) meromorphic extension to (at least) $\{\operatorname{Re} s > D - \alpha\}$. The set of all (visible) complex dimensions of A (i.e., the poles of $\tilde{\zeta}_A$) is given by

$$\mathcal{P}(\tilde{\zeta}_A) = \left\{ s_k = D + \frac{2\pi}{T} \mathrm{i}k : \hat{G}_0(\frac{k}{T}) \neq 0,\ k \in \mathbb{Z} \right\};$$

Minkowski Nonmeasurable Sets

Theorem (. . . continued)

they are all simple. Here, $\hat{G}_0(s) := \int_0^T \mathrm{e}^{-2\pi \mathrm{i} s\cdot\tau} G(\tau)\,\mathrm{d}\tau$.
For all $s_k \in \mathcal{P}(\tilde{\zeta}_A)$, $\operatorname{res}(\tilde{\zeta}_A, s_k) = \frac{1}{T}\hat{G}_0(\frac{k}{T})$. *We have*

$$|\operatorname{res}(\tilde{\zeta}_A, s_k)| \leq \frac{1}{T}\int_0^T G(\tau)\,\mathrm{d}\tau, \qquad \lim_{k\to\pm\infty} \operatorname{res}(\tilde{\zeta}_A, s_k) = 0.$$

Moreover,

$$\operatorname{res}(\tilde{\zeta}_A, D) = \frac{1}{T}\int_0^T G(\tau)\,\mathrm{d}\tau$$

and

$$\mathcal{M}_*^D(A) < \operatorname{res}(\tilde{\zeta}_A, D) < \mathcal{M}^{*D}(A).$$

In particular, A is not Minkowski measurable.

Remarks:

- Under the assumptions of the theorem, the *average Minkowski content* $\widetilde{\mathcal{M}}$ *of* A (defined as a suitable Cesaro logarithmic average of $|A_\varepsilon|/\varepsilon^{N-D}$) *exists and is given by*

$$\operatorname{res}(\tilde{\zeta}_A, D) = \widetilde{\mathcal{M}} = \frac{1}{T}\int_0^T G(\tau)d\tau.$$

- Provided $D < N$, an entirely analogous theorem holds for ζ_A (instead of $\tilde{\zeta}_A$), the distance zeta function of A, except for the fact that the residues take different values. In particular,

$$\operatorname{res}(\zeta_A, D) = (N-D)\widetilde{\mathcal{M}}.$$

Minkowski Nonmeasurable Sets

Example

If A is the ternary Cantor set, we have (see [L-vF, 2000])

$$|A_t| = t^{1-D} G(\log t^{-1}) \text{ as } t \to 0^+,$$

where $D = \log_3 2$ and the nonconstant function G is $\log 3$-periodic:

$$G(\tau) = 2^{1-D} \left(2^{\{\frac{\tau - \log 2}{\log 3}\}} + \left(\frac{3}{2}\right)^{-\{\frac{\tau - \log 2}{\log 3}\}} \right),$$

where $\{x\} := x - \lfloor x \rfloor$ is the fractional part of x and $\lfloor x \rfloor$ is the integer part of x.

Conclusion: $\tilde{\zeta}_A$ and ζ_A have a (unique) meromorphic extension to $\{\operatorname{Re} s > D - \alpha\}$ for any $\alpha > 0$, and hence to all of $\mathbb{C}$.

Definition

The principal complex dimensions of a bounded set A in $\mathbb{R}^N$ are given by

$$\mathcal{P}(\zeta_A) := \dim_{\mathbb{C}} A \cap \{\operatorname{Re} s = D(\zeta_A)\},$$

where $\dim_{\mathbb{C}} A$ denotes the set of (visible) complex dimensions of A. (Recall that $D(\zeta_A) = D(\tilde{\zeta}_A) = \overline{\dim}_B A$.) The vertical line $\{\operatorname{Re} s = D(\zeta_A)\}$ is called the critical line.

Remark: For the ternary Cantor set A, we have

$$\mathcal{P}(\zeta_A) = \dim_{\mathbb{C}} A = \log_3 2 + \frac{2\pi}{\log 3} \mathrm{i}\mathbb{Z}.$$

Distance Zeta Functions of Relative Fractal Drums

Definition

A relative fractal drum is a pair (A, Ω) of nonempty subsets A and Ω (open subset) of $\mathbb{R}^N$, such that $|\Omega| < \infty$ and there exists $\delta > 0$ such that $\Omega \subset A_\delta$.

Note that A and Ω may be unbounded. We do not assume $A \subseteq \overline{\Omega}$.

Definition

Let $t \in \mathbb{R}$. Then the *upper t-dimensional Minkowski content* of A relative to Ω is given by

$$\mathcal{M}^{*t}(A, \Omega) = \overline{\lim}_{\varepsilon \to 0^+} \frac{|A_\varepsilon \cap \Omega|}{\varepsilon^{N-t}}.$$

Definition

The *upper box dimension* of the relative fractal drum (A, Ω) is given by

$$\overline{\dim}_B(A, \Omega) = \inf\{t \in \mathbb{R} : \mathcal{M}^{*t}(A, \Omega) = 0\}.$$

It may be negative, and even equal to $-\infty$; this is related to the flatness of (A, Ω). (This latter concept is not discussed here; see [LRŽ] for details.)

Analyticity of Relative Zeta Functions

Definition (Relative distance zeta function of (A, Ω), LRŽ)

$$\zeta_A(s, \Omega) = \int_\Omega d(x, A)^{s-N} \mathrm{d}x,$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large.

Theorem (LRŽ)

- *$\zeta_A(s, \Omega)$ is holomorphic for $\operatorname{Re} s > \overline{\dim}_B(A, \Omega)$; the lower bound $\operatorname{Re} s > \overline{\dim}_B(A, \Omega)$ is optimal. Hence, the abscissa of convergence of $\zeta_A(\cdot, \Omega)$ is equal to $\overline{\dim}_B(A, \Omega)$, the relative upper box dimension of (A, Ω).*
- *Assume that $D = \dim_B(A, \Omega)$ exists and $\mathcal{M}_*^D(A, \Omega) > 0$. Then, if $s \in \mathbb{R}$ and $s \to D^+$, we have $\zeta_A(s, \Omega) \to +\infty$.*

Definition (Relative tube zeta function of (A, Ω), LRŽ)

$$\tilde{\zeta}_A(s, \Omega) = \int_0^\delta t^{s-N-1} |A_t \cap \Omega| dt,$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. (Here, $\delta > 0$ is fixed.)

Remark: The above theorem is valid without change for $\tilde{\zeta}_A(\cdot, \Omega)$ (instead of $\zeta_A(\cdot, \Omega)$). In particular,

$$\overline{\dim}_B(A, \Omega) = \text{the abscissa of convergence of } \tilde{\zeta}_A(\cdot, \Omega).$$

Example (1)

If $A = \{(x, y) : y = x^\alpha,\ 0 < x < 1\}$ with $\alpha \in (0, 1)$ and $\Omega = (-1, 0) \times (0, 1)$, then $\dim_B(A, \Omega) = 1 - \alpha$, which is < 1. Note that A and $\overline{\Omega}$ are disjoint.

Example (2)

If $A = \{(0, 0)\}$ (the origin in $\mathbb{R}^2$) and $\Omega = \{(x, y) \in (0, 1) \times \mathbb{R} : 0 < y < x^\alpha\}$ with $\alpha > 1$, then $\dim_B(A, \Omega) = 1 - \alpha$, which is < 0. Note that Ω is flat at A.

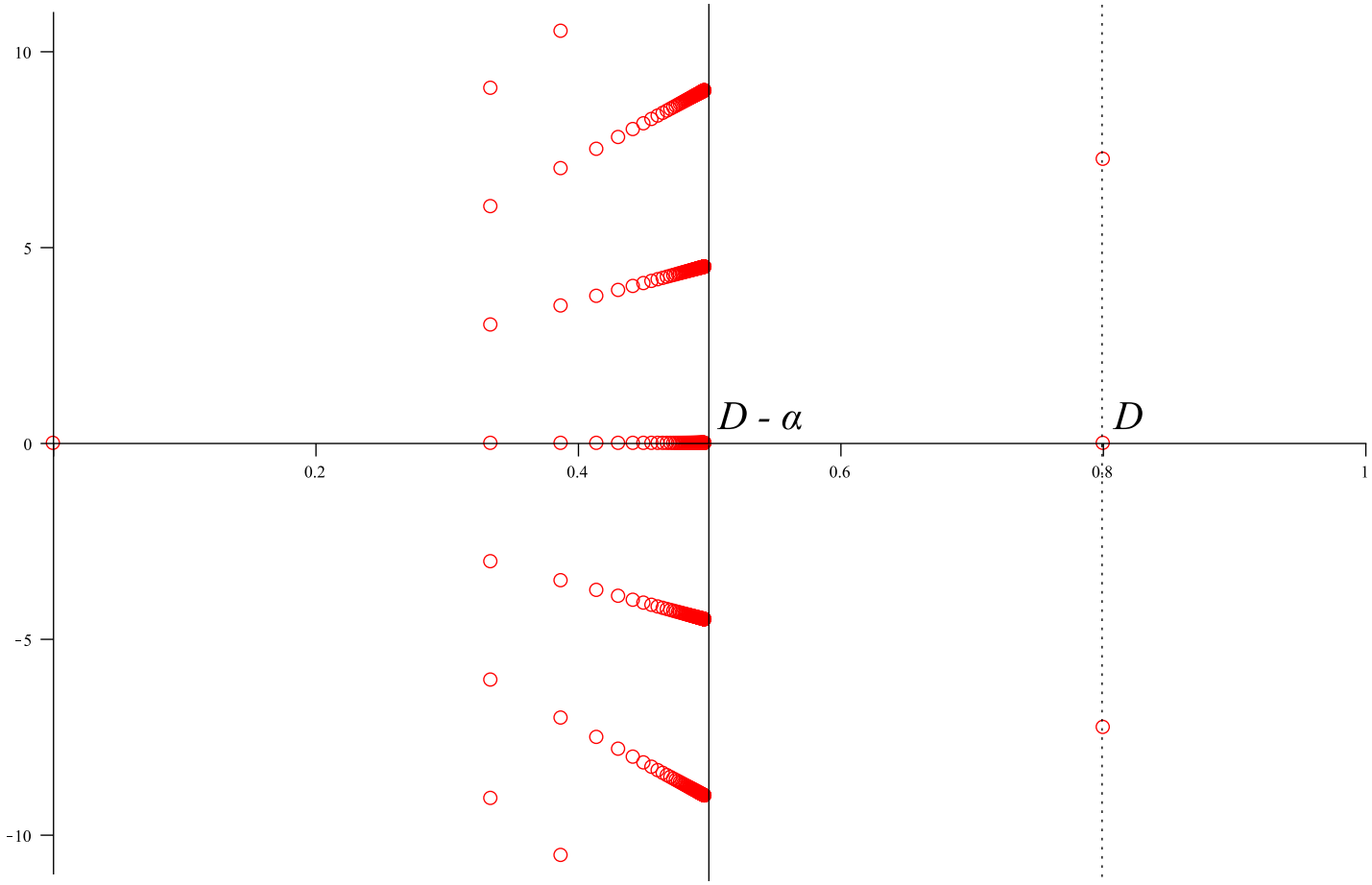


Figure 11: The complex dimensions of a relative fractal drum.

In the previous figure (Fig.), the set of complex dimensions $\mathcal{D}$ of the relative fractal drum (A, Ω), obtained as a union of relative fractal drums $\{(A_j, \Omega_j)\}_{j=1}^{\infty}$ involving generalized Cantor sets. Here, $D = 4/5$ and $\alpha = 3/10$.

Furthermore, $D(\tilde{\zeta}_A(\cdot, \Omega)) = 4/5, D_{mer}(\tilde{\zeta}_A(\cdot, \Omega)) = D - \alpha = 1/2$ and $2^{-1} + 4\pi(\log 2)i\mathbb{Z}$ is the set of nonisolated singularities of $\tilde{\zeta}_A(\cdot, \Omega)$.

The set $\mathcal{D}$ is contained in a union of countably many rays emanating from the origin. The dotted vertical line is the holomorphy critical line $\{\operatorname{Re} s = D\}$ of $\tilde{\zeta}_A(\cdot, \Omega)$, and to the left of it is the meromorphy critical line $\{\operatorname{Re} s = D - \alpha\}$.

Definition

A *function* $G : \mathbb{R} \to \mathbb{R}$ is *transcendentally quasiperiodic* of infinite order (resp., of finite order m) if it is of the form

$$G(\tau) = H(\tau, \tau, \ldots),$$

where $H : \mathbb{R}^\infty \to \mathbb{R}$ (resp., $H : \mathbb{R}^m \to \mathbb{R}$) is a function which is T_j-periodic in its j-th component, for each $j \in \mathbb{N}$ (resp., for each $j = 1, \cdots, m$), with $T_j > 0$ as minimal periods, and such that the set of *quasiperiods* $\{T_j : j \geq 1\}$ (resp., $\{T_j : j = 1, \cdots, m\}$) is *algebraically independent*; i.e., is independent over the ring of algebraic numbers.

Definition

A *relative fractal drum* (A, Ω) in $\mathbb{R}^N$ is said to be transcendentally *quasiperiodic* if

$$|A_t \cap \Omega| = t^{N-D}(G(\log(1/t)) + o(1)) \text{ as } t \to 0^+,$$

where the function G is transcendentally quasiperiodic.

In the special case where $A \subseteq \mathbb{R}^N$ is bounded and $\Omega = \mathbb{R}^N$, then the *set* A is said to be transcendentally *quasiperiodic.*

Hyperfractals

Using Alan Baker's theorem (in the theory of transcendental numbers) and generalized Cantor sets $C^{(m,a)}$ with two parameters (as in the above example), *it is possible to construct a transcendentally quasiperiodic bounded set A in $\mathbb{R}$* with infinitely many algebraically independent quasiperiods.

For this set, we show that $\tilde{\zeta}_A(s)$ has *the critical line $\{\operatorname{Re} s = D\}$ as a natural boundary*, where $D = \dim_B A$. (This means that $\tilde{\zeta}_A(s)$ does not have a meromorphic extension to the left of $\{\operatorname{Re} s = D\}$.) Moreover, *all of the points of the critical line $\{\operatorname{Re} s = D\}$ are singularities of $\tilde{\zeta}_A(s)$*; the same is true for $\zeta_A(s)$ (instead of $\tilde{\zeta}_A(s)$). (See [LRŽ] for the detailed construction.)

The set A is then said to be (maximally) *hyperfractal*.

Remark

The above construction of maximally hyperfractal and transcendentally quasiperiodic sets (and relative fractal drums) of infinite order has been applied in [LRŽ] in different contexts. In particular, it has been applied to prove that certain estimates obtained by the first author and regarding the abscissae of meromorphic continuation of the spectral zeta function of fractal drums are *sharp*, in general.

This construction is also relevant to the definition of *fractality* given in terms of complex dimensions. Recall that in the theory of complex dimensions, an object is said to be "fractal" if it has at least one nonreal complex dimension (with positive real part) or else if the associated fractal zeta function has a natural boundary (along a suitable contour). This new higher-dimensional theory of complex dimensions now enables us to define fractality in full generality.

Future Research Directions

1. **Fractal tube formulas and geometric complex dimensions**
   a. Case of self-similar sets
   b. Devil's staircase (cf. the definition of fractality)
   c. Weierstrass function
   d. Julia sets and Mandelbrot set

Possible geometric interpretation: *fractal curvatures* (*even for complex dimensions*)
Connections with earlier joint work of the author with Erin Pearse and Steffen Winter for fractal sprays and self-similar tilings.

Added note: A general fractal tube formula has now been obtained by the authors of [LRŽ]. This formula has been applied to a number of self-similar and non self-similar examples, including the Sierpinski gasket and carpet as well as their higher-dimensional counterparts. See [LRŽ, Chapter 5] and the relevant references in the bibliography.

Future Research Directions

2. **Spectral complex dimensions** Determine the complex dimensions of a variety of fractal drums (via the associated spectral functions) and compare these spectral complex dimensions with the geometric complex dimensions discussed in (1) just above.

3. **Generalization to metric measure spaces or Ahlfors' spaces** (joint work in progress with Sean Watson)

Connections with nonsmooth geometric analysis and analysis on fractals

Future Research Directions

4. **Box-counting zeta functions**
(joint work in progress with John Rock and Darko Žubrinić)

5. **Spectral zeta functions of relative fractal drums (spectral complex dimensions)**
Connections with geometric fractal zeta functions and complex dimensions

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