An Introduction to the Theory of Complex Dimensions and Fractal Zeta Functions

Michel L. Lapidus

University of California, Riverside
Department of Mathematics

http://www.math.ucr.edu/~lapidus/
lapidus@math.ucr.edu

6th Cornell Conference on Analysis, Probability,
and Mathematical Physics on Fractals

Cornell University
Ithaca NY

June 13, 2017
**Figure 1:** The rain of complex dimensions falling from the music of the angel’s fractal harp (or fractal string).
1 Definitions and Motivations
   • Minkowski Content and Box Dimension
   • Singularities of Functions

2 Fractal strings
   • Zeta Functions of Fractal Strings

3 Distance and Tube Zeta Functions
   • Definition
   • Analyticity
   • Residues of Distance Zeta Functions
   • Residues of Tube Zeta Functions
   • \((\alpha, \beta)\)-chirps
   • Meromorphic Extensions of Fractal Zeta Functions

4 Relative Distance and Tube Zeta Functions

5 References
Goals

- Introducing a new class of **fractal zeta functions**: distance and tube zeta functions associated with bounded fractal sets in Euclidean spaces of arbitrary dimensions.
- Developing a **higher-dimensional theory of complex fractal dimensions** valid for arbitrary compact sets (and eventually, for suitable metric measure spaces).
- Merging of aspects of complex, spectral and harmonic analysis, geometry, and number theory of fractal sets in $\mathbb{R}^N$. 
Main References for this Talk:


(And nine related papers by the authors of [LRŽ]; see the bibliography.)

† of Utah Valley University, USA
‡ of the University of Zagreb, Croatia
Book in Preparation:

**Fractal Tube Formula:**

\[ \sum_{\omega \in D_A} c_\omega \frac{\varepsilon^{N-\omega}}{\varepsilon^{N-\omega}} \]

where

\[ c_\omega = \mu_\omega(\mathcal{J}_A, \omega) \]

\[ \mathcal{J}_A(s) = \int_{\mathbb{R}^N} d(x, A)^{s-N} dx \]
Set of complex dimension of $A$: $D_A = \{ \omega : \omega \text{ pole of } A \}$

$A_\varepsilon = \{ x \in \mathbb{R}^n : d(x, A) < \varepsilon \}$

$\varepsilon$-neighborhood of $A$.

$V(\varepsilon)_A = |A_\varepsilon| = n \cdot \text{dim}^\varepsilon = \text{volume of } A_\varepsilon.$
Example: (Cantor string) \( L = CS = \left( \frac{1}{3} \right)_j \)\
\[ l_1 = \frac{1}{3}, \quad l_2 = \frac{1}{3}, \quad l_3 = \frac{1}{5} \]

\[ 0, \quad \frac{1}{21}, \quad \frac{5}{21}, \quad \frac{13}{21}, \quad \frac{19}{21}, \quad \frac{25}{21} \]

\[ \sum_{n \geq 1} \frac{(2^n)^{-\omega_m}}{\omega_n (1 - \omega_n)} - 2 \varepsilon \]

where \( D \varepsilon = \{ \text{poles of} J_\lambda \} \)

\[ J_{CS}(s) = \sum_{j=1}^{\infty} \frac{\lambda_j^{-s}}{3^j} = \frac{\ln 2 + \frac{i \pi}{\ln 3} \text{ if } n \in \mathbb{Z}}{3^j} \]

\[ \text{D-dimension of CS} \]

Geometric zeta function of \( L \)

\[ \zeta_A(s) = \frac{2^{1-s}}{1-s} J_{CS}(s) \]
Definitions and Motivations

Fractal strings

Distance and Tube Zeta Functions

Relative Distance and Tube Zeta Functions

---

Example 2: Sierpinski gasket, \( A \cdot N = 2 \)

\[ \text{Exp 2: (Sierpinski gasket)} \]

\[ V_G(x) = x^2 - \text{D}G(\log x) + c \]

\[ S_A(s) = \frac{6\sqrt{3}}{2} \left( \frac{1}{s(2-s)} \right) \]

\[ D_S = \{ 0, 1 \} \]

\[ \omega_n = \lim_{n \to \infty} 3^m \]

---

\[ \text{Sierpinski Triangle} \]

\[ \text{Fractality and Unreality} \]

A geometric object is called a fractal if it has at least one non-real complex dimension.
The intuition behind the notion of complex dimension

\[ \omega = \alpha + i \beta \]

**Complex Dimension**

- **Real Part**
- **Imaginary Part**

**Amplitudes of the (Geometric) Waves**

**Frequencies of the (Geometric) Waves**
Figure 2: Stalagmites and stalactites in a fractal cave
Figure 3: Other fractal stalagmites
Minkowski Content

Let \( A \subset \mathbb{R}^N \) be a nonempty bounded set.

- \( \varepsilon \)-neighborhood of \( A \),

\[
A_\varepsilon = \{ y \in \mathbb{R}^N : d(y, A) < \varepsilon \}.
\]

- **Lower \( s \)-dimensional Minkowski content** of \( A \), \( s \geq 0 \):

\[
\mathcal{M}_{s}^*(A) := \liminf_{\varepsilon \to 0^+} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},
\]

where \( |A_\varepsilon| \) is the \( N \)-dimensional Lebesgue measure of \( A_\varepsilon \).

- **Upper \( s \)-dimensional Minkowski content** of \( A \):

\[
\mathcal{M}_s^*(A) := \limsup_{\varepsilon \to 0^+} \frac{|A_\varepsilon|}{\varepsilon^{N-s}}.
\]
**Box Dimensions**

- **Lower box dimension** of $A$: \( \dim_B A = \inf \{ s \geq 0 : \mathcal{M}^s(A) = 0 \} \).
- **Upper box dimension** of $A$:
  \[
  \overline{\dim}_B A = \inf \{ s \geq 0 : \mathcal{M}^s(A) = 0 \}.
  \]
- \( \dim_H A \leq \underline{\dim}_B A \leq \dim_B A \leq N \),
  where \( \dim_H A \) denotes the Hausdorff dimension of $A$.
- If \( \underline{\dim}_B A = \overline{\dim}_B A \), we write \( \dim_B A \), the box dimension of $A$.

- If there is \( d \geq 0 \) such that
  \[
  0 < \mathcal{M}^d_*(A) \leq \mathcal{M}^{*d}(A) < \infty,
  \]
  we say $A$ is **Minkowski nondegenerate**. Clearly, \( d = \dim_B A \).
- If \( |A_\varepsilon| \asymp \varepsilon^\sigma \) for all sufficiently small $\varepsilon$ and some $\sigma \leq N$, then $A$ is Minkowski nondegenerate and
  \[
  \dim_B A = N - \sigma.
  \]
Minkowski Content and Box Dimension

Minkowski Measurable and Nondegenerate Sets

If $\mathcal{M}_s^*(A) = \mathcal{M}^{*s}(A)$ for some $s$, we denote this common value by $\mathcal{M}_s^*(A)$ and call it the \textit{s-dimensional Minkowski content} of $A$.

Furthermore, if $\mathcal{M}^d(A) \in (0, \infty)$ for some $d \geq 0$, then $A$ is said to be \textit{Minkowski measurable}, with Minkowski content $\mathcal{M}^d(A)$ (often simply denoted by $\mathcal{M}$). Clearly, we then have $d = \dim_B A$.

**Example (The Cantor set)**

The triadic Cantor set $A$ has box dimension $\dim_B A = \log 2 / \log 3$. $A$ is \textit{not} Minkowski measurable (Lapidus & Pomerance, 1993).

**Example (a-set)**

Let $A = \{k^{-a} : k \in \mathbb{N}\}$ be the \textit{a-set}, for $a > 0$. Then $\dim_B A = 1/(1 + a)$ and $A$ is Minkowski measurable (L., 1991).
The Triadic Cantor Set

\[
\frac{|A_\varepsilon|}{\varepsilon^{1-d}} = G(\log_3 \varepsilon^{-1}) + O(\varepsilon^d) \text{ as } \varepsilon \to 0^+. \text{ Here, } d = \log_3 2.
\]

Figure 4: The oscillating nature of the function \( \varepsilon \mapsto |A_\varepsilon|/\varepsilon^{1-d} \) near \( \varepsilon = 0 \) for the triadic Cantor set \( A \), with \( d := \dim_B A = \log_3 2 \). Then, \( A \) is Minkowski nondegenerate, but is not Minkowski measurable [Lapidus & Pomerance, 1993]. The function \( G(\tau) \) is \( \log 3 \)-periodic. (See [Lapidus & van Frankenhuijsen, 2000, 2006 & 2013] for much more detailed information.)
Singular Function Generated by the Cantor Set

\[ y = d(x, A) \]

**Figure 5:** The graph of the distance function \( x \mapsto d(x, A) \), where \( A \) is the classic ternary (or triadic) Cantor set \( C^{(1/3)} \).
**Singular Function Generated by the Cantor Set**

Figure 6: For the triadic Cantor set $A$, the function $y = d(x, A)^{-\gamma}$, $x \in (0, 1)$, is Lebesgue integrable if and only if $\gamma < 1 - \log 2 / \log 3$. Here, $\dim_H A = \dim_B A = \log 2 / \log 3$, where $\dim_H A$ denotes the Hausdorff dimension of $A$. 
The Sierpinski carpet $A$ (two iterations are shown);

$$\dim_H A = \dim_B A = \frac{\log 8}{\log 3},$$

*A is Minkowski nondegenerate, but not Minkowski measurable* (L., 1993).
Figure 7: The classic self-similar Sierpinski carpet
(a) The Hölder case  
(b) The Lipschitz case

**Figure 8: The Sierpinski stalagmites** The graph of $f(x) = d(x, A)^r$, where $r \in (0, 1)$ or $r \geq 1$, respectively. Here, $r = 0.5$ (a) or $r = 1.3$ (b).
Figure 9: Fractal stalagmites associated with the Sierpinski carpet.
The figure on the previous slide depicts the graph of the distance function $y = d(x, A)$, defined on the unit square, where $A$ is the Sierpinski carpet. Only the first three generations of the countable family of pyramidal tents (called *stalagmites*) are shown.
**Figure 10:** Fractal stalactites associated with the Sierpinski carpet.
The figure on the previous slide shows the graph of the function $y = d(x, A)^{-\gamma}$, defined on the unit square, where $A$ is the Sierpinski carpet. Since $A$ is known to be Minkowski nondegenerate, this function is Lebesgue integrable if and only if $\gamma \in (-\infty, 2 - D)$, $D = \dim_B A = \log_3 8$. For $\gamma > 0$, the graph consists of countably many connected components, called *stalactites*, all of which are unbounded.
Zeta Functions of Fractal Strings

Definition of Fractal Strings

- \( \mathcal{L} = (\ell_j)_{j \geq 1} \) a fractal string (L., 1991, L. & Pomerance, 1993): a nonincreasing sequence of positive numbers \((\ell_j)\) such that \(\sum_j \ell_j < \infty\). [Alternatively, \(\mathcal{L}\) can be viewed as a sequence of scales or as the lengths of the connected components (open intervals) of a bounded open set \(\Omega \subset \mathbb{R}\).]

- The zeta function (geometric zeta function) of the fractal string \(\mathcal{L}\) is the Dirichlet series:

  \[
  \zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} (\ell_j)^s,
  \]

  for \(s \in \mathbb{C}\) with \(\text{Re}\ s\) large enough.
Consider the open intervals $I_j = (a_j, a_{j-1})$ for $j \geq 1$, where

$$a_j := \sum_{k>j} \ell_k, \quad \text{and} \quad \ell_j := |I_j|.$$ 

Define $A = \{a_j\}$. Then $A$ is a bounded set, $A \subset \mathbb{R}$, and $a_j \to 0$ as $j \to \infty$. The set $A = A_L$ is introduced in [LR\v{Z}].

$\text{dim}_B A$ is defined via the upper Minkowski content of $A$, as usual.
Theorem (L)

The abscissa of convergence of $\zeta_L$ is equal to $\overline{\dim}_B A$:

$$\overline{\dim}_B A = \inf \left\{ \alpha > 0 : \sum_{j=1}^{\infty} (\ell_j)^\alpha < \infty \right\}.$$ 

Theorem (L, L-vF)

- $\zeta_L(s)$ is holomorphic on the right half-plane $\{\text{Re} s > \overline{\dim}_B A\}$;
- The lower bound $\overline{\dim} B A$ is optimal, both from the point of view of the absolute convergence and the holomorphic continuation.
- Moreover, if $s \in \mathbb{R}$ and $s \to \overline{\dim} B A$ from the right, then $\zeta_L(s) \to +\infty$. 
Corollary

The abscissa of holomorphic continuation and the abscissa of (absolute) convergence of $\zeta_L$ both coincide with the (upper) Minkowski dimension of $\mathcal{L}$:

$$D_{\text{hol}}(\zeta_L) = D(\zeta_L) = \overline{\dim}_B A.$$

Remarks: In the above discussion, $A$ could be replaced by $\partial \Omega$, the boundary of any geometric realization of $\mathcal{L}$ by a bounded open set $\Omega \subset \mathbb{R}$. 
Let \( A \subset \mathbb{R}^N \) be an arbitrary bounded set, and let \( \delta > 0 \) be fixed. As before, \( A_\delta \) denotes the \( \delta \)-neighborhood of \( A \).

**Definition (L., 2009; LRŽ, 2013)**

The **distance zeta function** of \( A \) is defined by

\[
\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} \, dx,
\]

for \( s \in \mathbb{C} \) with \( \text{Re } s \) sufficiently large.
Remarks:

- For $s \in \mathbb{C}$ such that $\text{Re } s < N$, the function $d(x, A)^{s-N}$ is singular on $A$.

- The inequality $\delta < \delta_1$ implies that

$$\zeta_A(s; A_{\delta_1}) - \zeta_A(s) = \int_{A_{\delta, \delta_1}} d(x, A)^{s-N} \, dx$$

is an entire function. As a result, the definition of $\zeta_A = \zeta_A(\cdot, A_\delta)$ does not depend on $\delta$ in an essential way. In particular, the complex dimensions of $A$ (i.e., the poles of $\zeta_A$) do not depend on $\delta$. 
Zeta Function of the Set $A$ Associated to a Fractal String $L$

- Let $L = (\ell_j)$ be a fractal string, and $A = (a_j)$, $a_j = \sum_{k \geq j} \ell_k$.
- We would like to compare $\zeta_L(s) = \sum_j (\ell_j)^s$ and

$$
\zeta_A(s) = \int_{-\delta}^{1+\delta} d(x, A)^{s-1} dx, \text{ for } \delta \geq \ell_1/2.
$$

- The zeta functions of $L$ and $A$ are ‘equivalent’, $\zeta_A(s) \sim \zeta_L(s)$, in the following sense:

$$
\zeta_A(s) = a(s)\zeta_L(s) + b(s),
$$

where $a(s)$ and $b(s)$ are explicit functions which are holomorphic on $\{\Re s > 0\}$, and $a(s) \neq 0$ for all such $s$. It follows that (when they exist) the meromorphic extensions of $\zeta_A(s)$ and $\zeta_L(s)$ have the same sets of poles in $\{\Re s > 0\}$ (i.e., the same set of visible complex dimensions up to 0).
Definition

The complex dimensions of a fractal string $\mathcal{L}$ (L & vF, 1996) are defined as the poles of $\zeta_\mathcal{L}$.

Definition

The complex dimensions of a bounded set $A \subset \mathbb{R}^N$ are defined as the poles of $\zeta_A$ (L., 2009; LRŽ, 2013).

Remark: We assume here that the zeta functions involved have a meromorphic extension (necessarily unique, by the principle of analytic continuation) to some suitable region $U \subset \mathbb{C}$.

Visible complex dimensions: the poles of $\zeta_A$ in $U$. 
Holomorphy Half-Plane of the Distance Zeta Function

Let $A$ be a nonempty bounded set in $\mathbb{R}^N$; given $\delta$ a fixed positive number, let $\zeta_A(s) = \int_{A_\delta} d(x, A)^{s-N} dx$ as before.

**Theorem (LRŽ)**
- $\zeta_A(s)$ is holomorphic on the right half-plane $\{\text{Re} s > \overline{\text{dim}}_B A\}$; the lower bound $\overline{\text{dim}}_B A$ is optimal from the point of view of the convergence of the Lebesgue integral defining $\zeta_A$.
- Moreover, if $D = \text{dim}_B A$ exists, $D < N$, and $\mathcal{M}_*(A) > 0$, then $\zeta_A(s) \to +\infty$ as $s \in \mathbb{R}$ and $s \to D^+$; so that the lower bound $\overline{\text{dim}}_B A$ is also optimal from the point of view of the holomorphic continuation.

**Remark:** If $s \in \mathbb{R}$ and $s < \overline{\text{dim}}_B A$, then $\zeta_A(s) = +\infty$. 
Corollary (LRŽ)

The abscissa of (absolute) convergence of $\zeta_A$ is equal to $\dim_B A$, the (upper) Minkowski dimension of $A$:

$$D(\zeta_A) := \inf \left\{ \alpha \in \mathbb{R} : \int_{A_\delta} d(x, A)^{\alpha - N} dx < \infty \right\}$$

$$= \dim_B A.$$ 

Corollary (LRŽ)

Assume that $D = \dim_B A$ exists, $D < N$, and $\mathcal{M}_*^D(A) > 0$. Then the abscissa of holomorphic continuation and the abscissa of (absolute) convergence of $\zeta_A$ both coincide with the (upper) Minkowski dimension of $A$:

$$D_{\text{hol}}(\zeta_A) = D(\zeta_A) = \dim_B A.$$
Definition

The **abscissa of holomorphic continuation** of \( \zeta_A \) is given by

\[
D_{hol}(\zeta_A) := \inf\{\alpha \in \mathbb{R} : \zeta_A(s) \text{ is holomorphic on } \Re s > \alpha\}
\]

Furthermore, the open half-plane \( \{\Re s > D_{hol}(\zeta_A)\} \) is called the **holomorphy half-plane** of \( \zeta_A \), while \( \{\Re s > D(\zeta_A)\} \) is called the **half-plane of (absolute) convergence** of \( \zeta_A \).

**Remark**: In general, we have

\[
D_{hol}(\zeta_A) \leq D(\zeta_A) = \overline{\dim}_B A.
\]
We assume that $\zeta_A(s) = \zeta_A(s, A_\delta)$ can be meromorphically extended to a neighborhood of $D := \dim_B A$, and $D < N$. We write $\zeta_A$ or $\zeta_A(\cdot, A_\delta)$, interchangeably.

**Theorem (LRŽ)**

If $\mathcal{M}^D_*(A) < \infty$, then $s = D$ is a simple pole of $\zeta_A$ and

$$(N - D)\mathcal{M}^D_*(A) \leq \text{res}(\zeta_A(\cdot, A_\delta), D) \leq (N - D)\mathcal{M}^D_*(A).$$

The value of $\text{res}(\zeta_A(\cdot, A_\delta), D)$ does not depend on $\delta > 0$.

*Remark*: For the triadic Cantor set, we have strict inequalities.
Theorem (LRŽ)

If $A$ is Minkowski measurable (i.e., $\mathcal{M}^D(A)$ exists and $\mathcal{M}^D(A) \in (0, \infty)$), then

$$\text{res}(\zeta_A(\cdot, A_\delta), D) = (N - D)\mathcal{M}^D(A).$$
Let $A \subset \mathbb{R}^N$ be an arbitrary bounded set, and let $\delta > 0$ be fixed.

**Definition**

The *tube zeta function of $A$* associated with the tube function $t \mapsto |A_t|$, is given by (for some fixed, small $\delta > 0$)

$$\tilde{\zeta}_A(s) = \int_{0}^{\delta} t^{s-N-1}|A_t| \, dt,$$

for $s \in \mathbb{C}$ with $\text{Re} \, s$ sufficiently large.

*Remark*: The choice of $\delta$ is unimportant, from the point of view of the theory of complex dimensions. Indeed, changing $\delta$ amounts to adding an entire function to $\tilde{\zeta}_A$. 
The next result follows from its counterpart stated earlier for the distance zeta function $\zeta_A$ (see the sketch of the proof given below):

**Corollary (LRŽ)**

If $D = \dim_B A$ exists, $D < N$ and $\tilde{\zeta}_A$ has a meromorphic extension to a neighborhood of $s = D$, then

$$\mathcal{M}_*^D(A) \leq \text{res}(\tilde{\zeta}_A, D) \leq \mathcal{M}^{*D}(A).$$

In particular, if $A$ is Minkowski measurable, then

$$\text{res}(\tilde{\zeta}_A, D) = \mathcal{M}_*^D(A).$$
The proof of the previous corollary rests on the following identity, which is valid on \( \{ \text{Re } s > \overline{D} \} \), where \( \overline{D} = \overline{\text{dim}_BA} \):

\[
\zeta_A(s, A_\delta) = \delta^{s-N}|A_\delta| + (N - s)\tilde{\zeta}_A(s).
\]

**Remark:** It follows from the above equation that if \( D < N \), then \( \tilde{\zeta}_A \) has a meromorphic extension to a given domain \( U \subseteq \mathbb{C} \) iff \( \zeta_A \) does. In particular, \( \tilde{\zeta}_A \) and \( \zeta_A \) have the same (visible) complex dimensions; that is, the same poles within \( U \).
Residues of Fractal Zeta Functions of Generalized Cantor Sets

Example (1)

For the generalized Cantor sets $A = C^{(a)}$, $a \in (0, 1/2)$, we have $D(a) = \dim_B C^{(a)} = \log_{1/a} 2$. Moreover,

\[
\mathcal{M}_*^D(A) = \frac{1}{D} \left( \frac{2D}{1-D} \right)^{1-D},
\]

\[
\mathcal{M}^{*D}(A) = 2(1-a) \left( \frac{1}{2} - a \right)^{D-1},
\]

and

\[
\text{res}(\tilde{\zeta}_A, D) = \frac{2}{\log 2} \left( \frac{1}{2} - a \right)^D.
\]
Example (1 continued)

For all $a \in (0, 1/2)$, we have

$$M^D_*(A) < \text{res}(\tilde{\zeta}_A(s), D) < M^*D(A).$$

Also,

$$\text{res}(\zeta_A, D) = (1 - D) \text{res}(\tilde{\zeta}_A, D).$$

Remark: With this notation, the classic ternary Cantor set is just $C^{(1/3)}$. For any $a \in (0, 1/2)$, the generalized Cantor set $C^{(a)}$ is constructed in much the same way as $C^{(1/3)}$, by removing open “middle $a$-intervals” at each stage of the construction.
Example (2)

The $a$-string associated with $A := \{k^{-a} : k \in \mathbb{N}\}$, $a > 0$, is given by

$$\mathcal{L} = (\ell_j)_{j \geq 1}, \quad \ell_j = j^{-a} - (j + 1)^{-a}.$$  

We have:

$$D(a) = \dim_B A = \frac{1}{1 + a},$$

$$\text{res}(\tilde{\zeta}_A, D) = \mathcal{M}^D(A) = \frac{2^{1-D}}{D(1-D)} a^D,$$

$$\text{res}(\zeta_A, D) = (1 - D)\mathcal{M}^D(A) = \frac{2^{1-D} a^D}{D},$$

and

$$\text{res}(\zeta_{\mathcal{L}}, D) = a^D.$$
Definition (1)

Let $\alpha > 0$ and $\beta > 0$. The standard $(\alpha, \beta)$-chirp is the graph of $y = x^{\alpha} \sin x^{-\beta}$ near the origin (here $\alpha = 1/2$, $\beta = 1$):
Definition of the \((\alpha, \beta)\)-chirp

Let \(\alpha > 0\) and \(\beta > 0\). The geometric \((\alpha, \beta)\)-chirp is the following countable union of vertical intervals in the plane (‘approximation’ of the standard \((\alpha, \beta)\)-chirp):

\[
\Gamma(\alpha, \beta) = \bigcup_{k \in \mathbb{N}} \{k^{-1/\beta}\} \times (0, k^{-\alpha/\beta}).
\]
The distance zeta function of $\Gamma(\alpha, \beta)$ can be computed as follows:

$$\zeta_{\Gamma(\alpha, \beta)}(s) \sim \frac{1}{s - 1} \sum_{k=1}^{\infty} k^{-\frac{\alpha}{\beta}}(1 + \frac{1}{\beta})(s-1),$$

where, as before, we define

$$\zeta_A(s) \sim f(s) \iff f(s) = a(s)\zeta_A(s) + b(s),$$

with $a(s), b(s)$ holomorphic on $\{\text{Re} s > r\}$, for some $r < \dim_B A$, and $a(s) \neq 0$ for all such $s$.

The series converges iff $\text{Re} s > \max\{1, 2 - \frac{1+\alpha}{1+\beta}\}$; hence,

$$\dim_B \Gamma(\alpha, \beta) = \max\{1, 2 - \frac{1+\alpha}{1+\beta}\}.$$

This is the analog of Tricot’s formula (which was originally proved for the standard $(\alpha, \beta)$-chirp).
Theorem (LRŽ (Minkowski measurable case))

Given $A \subset \mathbb{R}^N$, assume that there exist $\alpha > 0$, $\mathcal{M} \in (0, \infty)$ and $D \geq 0$ such that the tube function $t \mapsto |A_t|$ satisfies

$$|A_t| = t^{N-D} (\mathcal{M} + O(t^\alpha)) \quad \text{as } t \to 0^+. $$

Then $A$ is Minkowski measurable, and we have:

$$\dim_B A = D, \quad \mathcal{M}^D(A) = \mathcal{M}, \quad \text{and } D(\tilde{\zeta}_A) = D. $$

Furthermore, $\tilde{\zeta}_A$ has a (unique) meromorphic extension to (at least) $\{\text{Re } s > D - \alpha\}$.

Moreover, the pole $s = D$ is unique, simple, and $\text{res}(\tilde{\zeta}_A, D) = \mathcal{M}$. 

Minkowski Measurable Sets
**Remark**: Provided $D < N$, the exact same results hold for $\zeta_A$, the distance zeta function of $A$. Then, we have instead

$$\text{res}(\zeta_A, D) = (N - D)M.$$
**Theorem (LRŽ (Minkowski nonmeasurable case))**

Given $A \subset \mathbb{R}^N$, assume that there exist $D \geq 0$, a nonconstant periodic function $G : \mathbb{R} \to \mathbb{R}$ with minimal period $T > 0$, and $\alpha > 0$, such that

$$|A_t| = t^{N-D} \left( G(\log t^{-1}) + O(t^\alpha) \right) \quad \text{as } t \to 0^+.$$  

Then we have:

$$\dim_B A = D, \quad \mathcal{M}^D_\ast(A) = \min G, \quad \mathcal{M}^{*D}_\ast(A) = \max G, \quad \text{and } D(\tilde{\zeta}_A) = D.$$  

Furthermore, $\tilde{\zeta}_A(s)$ has a (unique) meromorphic extension to (at least) $\{\Re s > D - \alpha\}$. The set of all (visible) complex dimensions of $A$ (i.e., the poles of $\tilde{\zeta}_A$) is given by

$$\mathcal{P}(\tilde{\zeta}_A) = \left\{ s_k = D + \frac{2\pi}{T} ik : \hat{G}_0\left(\frac{k}{T}\right) \neq 0, \ k \in \mathbb{Z} \right\}.$$
they are all simple. Here, \( \hat{G}_0(s) := \int_0^T e^{-2\pi is \cdot \tau} G(\tau) \, d\tau \).

For all \( s_k \in \mathcal{P}(\tilde{\zeta}_A) \), \( \text{res}(\tilde{\zeta}_A, s_k) = \frac{1}{T} \hat{G}_0(\frac{k}{T}) \). We have

\[
| \text{res}(\tilde{\zeta}_A, s_k) | \leq \frac{1}{T} \int_0^T G(\tau) \, d\tau, \quad \lim_{k \to \pm\infty} \text{res}(\tilde{\zeta}_A, s_k) = 0.
\]

Moreover,

\[
\text{res}(\tilde{\zeta}_A, D) = \frac{1}{T} \int_0^T G(\tau) \, d\tau
\]

and

\[
\mathcal{M}^D_*(A) < \text{res}(\tilde{\zeta}_A, D) < \mathcal{M}^*_D(A).
\]

In particular, \( A \) is not Minkowski measurable.
Remarks:

- Under the assumptions of the theorem, the *average Minkowski content* $\tilde{\mathcal{M}}$ of $A$ (defined as a suitable Cesaro logarithmic average of $|A_\varepsilon|/\varepsilon^{N-D}$) exists and is given by

$$\text{res}(\tilde{\zeta}_A, D) = \tilde{\mathcal{M}} = \frac{1}{T} \int_0^T G(\tau) d\tau.$$  

- Provided $D < N$, an entirely analogous theorem holds for $\zeta_A$ (instead of $\tilde{\zeta}_A$), the distance zeta function of $A$, except for the fact that the residues take different values. In particular,

$$\text{res}(\zeta_A, D) = (N - D)\tilde{\mathcal{M}}.$$
Example

If $A$ is the ternary Cantor set, we have (see [L-vF, 2000])

$$|A_t| = t^{1-D} G(\log t^{-1}) \text{ as } t \to 0^+,$$

where $D = \log_3 2$ and the nonconstant function $G$ is log 3-periodic:

$$G(\tau) = 2^{1-D} \left( 2^{\left\{ \frac{\tau - \log 2}{\log 3} \right\}} + \left( \frac{3}{2} \right)^{-\left\{ \frac{\tau - \log 2}{\log 3} \right\}} \right),$$

where $\{x\} := x - \lfloor x \rfloor$ is the fractional part of $x$ and $\lfloor x \rfloor$ is the integer part of $x$.

**Conclusion:** $\tilde{\zeta}_A$ and $\zeta_A$ have a (unique) meromorphic extension to $\{\Re s > D - \alpha\}$ for any $\alpha > 0$, and hence to all of $\mathbb{C}$. 
Definition

The **principal complex dimensions** of a bounded set $A$ in $\mathbb{R}^N$ are given by

$$\mathcal{P}(\zeta_A) := \dim_{\mathbb{C}} A \cap \{\Re s = D(\zeta_A)\},$$

where $\dim_{\mathbb{C}} A$ denotes the set of (visible) complex dimensions of $A$. (Recall that $D(\zeta_A) = D(\tilde{\zeta}_A) = \dim_B A$..) The vertical line $\{\Re s = D(\zeta_A)\}$ is called the **critical line**.

**Remark:** For the ternary Cantor set $A$, we have

$$\mathcal{P}(\zeta_A) = \dim_{\mathbb{C}} A = \log_3 2 + \frac{2\pi}{\log 3}i\mathbb{Z}.$$
Distance Zeta Functions of Relative Fractal Drums

**Definition**

A relative fractal drum is a pair \((A, \Omega)\) of nonempty subsets \(A\) and \(\Omega\) (open subset) of \(\mathbb{R}^N\), such that \(|\Omega| < \infty\) and there exists \(\delta > 0\) such that \(\Omega \subset A_\delta\).

Note that \(A\) and \(\Omega\) may be unbounded. We do not assume \(A \subseteq \overline{\Omega}\).

**Definition**

Let \(t \in \mathbb{R}\). Then the upper \(t\)-dimensional Minkowski content of \(A\) relative to \(\Omega\) is given by

\[
\mathcal{M}^t(A, \Omega) = \lim_{\varepsilon \to 0^+} \frac{|A_\varepsilon \cap \Omega|}{\varepsilon^{N-t}}.
\]
Definition

The *upper box dimension* of the relative fractal drum \((A, \Omega)\) is given by

\[
\dim_B(A, \Omega) = \inf \{ t \in \mathbb{R} : \mathcal{M}^t(A, \Omega) = 0 \}.
\]

It may be negative, and even equal to \(-\infty\); this is related to the flatness of \((A, \Omega)\). (This latter concept is not discussed here; see [LRŽ] for details.)
Analyticity of Relative Zeta Functions

**Definition (Relative distance zeta function of \((A, \Omega), \text{LRŽ}\)**

\[
\zeta_A(s, \Omega) = \int_{\Omega} d(x, A)^{s-N} \, dx,
\]

for \(s \in \mathbb{C}\) with \(\text{Re } s\) sufficiently large.

**Theorem (LRŽ)**

- \(\zeta_A(s, \Omega)\) is holomorphic for \(\text{Re } s > \text{dim}_B(A, \Omega)\); the lower bound \(\text{Re } s > \text{dim}_B(A, \Omega)\) is optimal. Hence, the abscissa of convergence of \(\zeta_A(\cdot, \Omega)\) is equal to \(\text{dim}_B(A, \Omega)\), the relative upper box dimension of \((A, \Omega)\).

- Assume that \(D = \text{dim}_B(A, \Omega)\) exists and \(\mathcal{M}^D_*(A, \Omega) > 0\). Then, if \(s \in \mathbb{R}\) and \(s \to D^+\), we have \(\zeta_A(s, \Omega) \to +\infty\).
Definition (Relative tube zeta function of \((A, \Omega), \text{LRŽ}\))

\[
\tilde{\zeta}_A(s, \Omega) = \int_0^\delta t^{s-N-1} |A_t \cap \Omega| dt,
\]

for \(s \in \mathbb{C}\) with \(\text{Re } s\) sufficiently large. (Here, \(\delta > 0\) is fixed.)

Remark: The above theorem is valid without change for \(\tilde{\zeta}_A(\cdot, \Omega)\) (instead of \(\zeta_A(\cdot, \Omega)\)). In particular,

\[
\overline{\text{dim}}_B(A, \Omega) = \text{the abscissa of convergence of } \tilde{\zeta}_A(\cdot, \Omega).
\]
**Example (1)**

If $A = \{(x, y) : y = x^\alpha, \ 0 < x < 1\}$ with $\alpha \in (0, 1)$ and $\Omega = (-1, 0) \times (0, 1)$, then $\dim_B(A, \Omega) = 1 - \alpha$, which is $< 1$. Note that $A$ and $\Omega$ are disjoint.

**Example (2)**

If $A = \{(0, 0)\}$ (the origin in $\mathbb{R}^2$) and $\Omega = \{(x, y) \in (0, 1) \times \mathbb{R} : 0 < y < x^\alpha\}$ with $\alpha > 1$, then $\dim_B(A, \Omega) = 1 - \alpha$, which is $< 0$. Note that $\Omega$ is flat at $A$. 
Figure 11: The complex dimensions of a relative fractal drum.
In the previous figure (Fig.), the set of complex dimensions $D$ of the relative fractal drum $(A, \Omega)$, obtained as a union of relative fractal drums $\{(A_j, \Omega_j)\}_{j=1}^{\infty}$ involving generalized Cantor sets. Here, $D = 4/5$ and $\alpha = 3/10$.

Furthermore, $D(\tilde{\zeta}_A(\cdot, \Omega)) = 4/5$, $D_{\text{mer}}(\tilde{\zeta}_A(\cdot, \Omega)) = D - \alpha = 1/2$ and $2^{-1} + 4\pi(\log 2)i\mathbb{Z}$ is the set of nonisolated singularities of $\tilde{\zeta}_A(\cdot, \Omega)$.

The set $D$ is contained in a union of countably many rays emanating from the origin. The dotted vertical line is the holomorphy critical line $\{\text{Re} \ s = D\}$ of $\tilde{\zeta}_A(\cdot, \Omega)$, and to the left of it is the meromorphy critical line $\{\text{Re} \ s = D - \alpha\}$. 
Definition

A function $G : \mathbb{R} \rightarrow \mathbb{R}$ is *transcendently quasiperiodic* of infinite order (resp., of finite order $m$) if it is of the form

$$G(\tau) = H(\tau, \tau, \ldots),$$

where $H : \mathbb{R}^{\infty} \rightarrow \mathbb{R}$ (resp., $H : \mathbb{R}^{m} \rightarrow \mathbb{R}$) is a function which is $T_j$-periodic in its $j$-th component, for each $j \in \mathbb{N}$ (resp., for each $j = 1, \cdots, m$), with $T_j > 0$ as minimal periods, and such that the set of *quasiperiods* $\{T_j : j \geq 1\}$ (resp., $\{T_j : j = 1, \cdots, m\}$) is *algebraically independent*; i.e., is independent over the ring of algebraic numbers.
A relative fractal drum \((A, \Omega)\) in \(\mathbb{R}^N\) is said to be transcendentally quasiperiodic if

\[ |A_t \cap \Omega| = t^{N-D}(G(\log(1/t)) + o(1)) \text{ as } t \to 0^+, \]

where the function \(G\) is transcendentally quasiperiodic.

In the special case where \(A \subseteq \mathbb{R}^N\) is bounded and \(\Omega = \mathbb{R}^N\), then the set \(A\) is said to be transcendentally quasiperiodic.
Using Alan Baker’s theorem (in the theory of transcendental numbers) and generalized Cantor sets $C^{(m,a)}$ with two parameters (as in the above example), it is possible to construct a transcendentally quasiperiodic bounded set $A$ in $\mathbb{R}$ with infinitely many algebraically independent quasiperiods.

For this set, we show that $\tilde{\varsigma}_A(s)$ has the critical line $\{\text{Re } s = D\}$ as a natural boundary, where $D = \dim_B A$. (This means that $\tilde{\varsigma}_A(s)$ does not have a meromorphic extension to the left of $\{\text{Re } s = D\}$.) Moreover, all of the points of the critical line $\{\text{Re } s = D\}$ are singularities of $\tilde{\varsigma}_A(s)$; the same is true for $\varsigma_A(s)$ (instead of $\tilde{\varsigma}_A(s)$). (See [LRŽ] for the detailed construction.)

The set $A$ is then said to be (maximally) hyperfractal.
Remark

The above construction of maximally hyperfractal and transcendently quasiperiodic sets (and relative fractal drums) of infinite order has been applied in [LRŽ] in different contexts. In particular, it has been applied to prove that certain estimates obtained by the first author and regarding the abscissae of meromorphic continuation of the spectral zeta function of fractal drums are *sharp*, in general.

This construction is also relevant to the definition of *fractality* given in terms of complex dimensions. Recall that in the theory of complex dimensions, an object is said to be “fractal” if it has at least one nonreal complex dimension (with positive real part) or else if the associated fractal zeta function has a natural boundary (along a suitable contour). This new higher-dimensional theory of complex dimensions now enables us to define fractality in full generality.
Future Research Directions

1. Fractal tube formulas and geometric complex dimensions
   a. Case of self-similar sets
   b. Devil’s staircase (cf. the definition of fractality)
   c. Weierstrass function
   d. Julia sets and Mandelbrot set

*Possible geometric interpretation: fractal curvatures (even for complex dimensions)*

*Connections with earlier joint work of the author with Erin Pearse and Steffen Winter for fractal sprays and self-similar tilings.*

**Added note:** A general fractal tube formula has now been obtained by the authors of [LRŽ]. This formula has been applied to a number of self-similar and non self-similar examples, including the Sierpinski gasket and carpet as well as their higher-dimensional counterparts. See [LRŽ, Chapter 5] and the relevant references in the bibliography.
Future Research Directions

2. **Spectral complex dimensions** Determine the complex dimensions of a variety of fractal drums (via the associated spectral functions) and compare these spectral complex dimensions with the geometric complex dimensions discussed in (1) just above.

3. **Generalization to metric measure spaces or Ahlfors’ spaces** (joint work in progress with Sean Watson)

   *Connections with nonsmooth geometric analysis and analysis on fractals*
Future Research Directions

4. **Box-counting zeta functions**
   (joint work in progress with John Rock and Darko Žubrinić)

5. **Spectral zeta functions of relative fractal drums (spectral complex dimensions)**
   
   *Connections with geometric fractal zeta functions and complex dimensions*
References


