

Group actions, the Mattila integral and continuous sum-product problems

Bochen Liu

University of Rochester

Acknowledgment: This work is done when the author was a research assistant in the Chinese University of Hong Kong, supported by Professor Ka-Sing Lau.

Falconer distance problem and Mattila integral

Given $E \subset \mathbb{R}^d$, $d \geq 2$, denote the set of distances generated by E as

$$\Delta(E) = \{|x - y| : x, y \in E\}.$$

In 1985, Falconer made the following conjecture.

Conjecture (Falconer distance conjecture)

For any $E \subset \mathbb{R}^d$, $d \geq 2$ $\dim_{\mathcal{H}}(E) > \frac{d}{2}$, its distance set $\Delta(E)$ has positive Lebesgue measure.

Falconer also gives examples to show that $d \geq 2$ is necessary and $\frac{d}{2}$ is the best dimensional threshold. We will discuss about his counterexamples later.

Falconer distance problem and Mattila integral

The best known results are due to Wolff (1999) for $d = 2$ and Erdogan (2005) for $d \geq 3$. They proved that $|\Delta(E)| > 0$ if $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}$. The main tool they used is the following theorem due to Mattila.

Theorem (Mattila integral, 1987)

Let μ be a Borel measure on E and define ν on $\Delta(E)$ by

$$\int f d\nu = \iint f(|x - y|) d\mu(x) d\mu(y),$$

i.e., $\nu = \Phi_*(\mu \times \mu)$ where $\Phi(x, y) = |x - y|$. Then $\hat{\nu} \in L^2(\mathbb{R})$ if and only if

$$\mathcal{M}(\mu) = \int \left(\int_{S^{d-1}} |\hat{\mu}(r\omega)|^2 d\omega \right)^2 r^{d-1} dr < \infty.$$

In particular, in this case $\Delta(E)$ has positive Lebesgue measure.

Falconer distance problem is in fact a continuous analog of the famous Erdős distance problem, which says

Conjecture (Erdős distance conjecture, 1946)

Let $P \subset \mathbb{R}^d$ be a set of N points, then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\#(\Delta(P)) \geq C_\epsilon N^{\frac{2}{d}-\epsilon}.$$

This conjecture has been solved by Guth and Katz (2010) on the plane, while it is still open in higher dimensions.

Erdős distance problem and group actions

In Guth-Katz argument, there is a very important observation which is originally due to Elekes and Sharir. Notice that for any quadruplet $x^1, x^2, y^1, y^2 \in \mathbb{R}^d$,

$$|x^1 - x^2| = |y^1 - y^2|$$

if and only if there exists $\theta \in O(d)$ such that

$$x^1 - \theta y^1 = x^2 - \theta y^2.$$

So we can work on $O(d)$ to estimate of “repetition” of the distances.

Since Falconer distance problem is a continuous analog of Erdős distance problem, a natural question at this point is, does this idea of groups actions help on Falconer distance problem?

Generalized Mattila integral

In 2015, Greenleaf, Iosevich, B.L. and Palsson observed that, the Mattila integral can be interpreted as

$$\int_0^\infty \left(\int_{S^{d-1}} |\hat{\mu}(r\omega)|^2 d\omega \right)^2 r^{d-1} dr = c_d \int |\hat{\mu}(\xi)|^2 \left(\int_{O(d)} |\hat{\mu}(\theta\xi)|^2 d\theta \right) d\xi.$$

We also generalized the classical Mattila integral to study the set of side lengths of k -simplex set of E , e.g., when $k = 2$, we may ask when

$$T_2(E) = \{ (|x^1 - x^2|, |x^2 - x^3|, |x^3 - x^1|) \in \mathbb{R}^3 : x^i \in E \}$$

has positive 3-dimensional Lebesgue measure.

Both Mattila's proof and GilP's proof of the Mattila integral are complicated and only valid on "distances". In this talk we shall prove a more general result whose proof is, however, very simple.

A new derivation of the Mattila integral

For simplicity, let's just consider the distance problem. Since $|x - y| = |x' - y'|$ if and only if there exists $g \in E(d)$, such that $gx = x'$, $gy = y'$, we have two ways to define ν on the distance set,

$$\nu(t) = \int_{|x-y|=t} \mu(x) \mu(y) d\sigma_t(x, y),$$

and, since $\{(x, y) : |x - y| = t\} = \{(gx_t, gy_t) : g \in E(d)\}$ is an orbit,

$$\nu(t) \approx \int_{E(d)} \mu(gx_t) \mu(gy_t) dg,$$

where $|x_t - y_t| = t$, arbitrary but fixed. Therefore

$$\int |\nu(t)|^2 \approx \int \left(\int_{|x-y|=t} \mu(x) \mu(y) \left(\int_{E(d)} \mu(gx_t) \mu(gy_t) dg \right) d\sigma_t(x, y) \right) dt$$

A new derivation of the Mattila integral

Since $|x - y| = t$ and the integral is independent on the choice of (x_t, y_t) , we can take $x_t = x$, $y_t = y$. Then by co-area formula (change of variables), it becomes

$$\int_{E(d)} \left| \int \mu(x) \mu(gx) dx \right|^2 dg = \int_{O(d)} \int_{\mathbb{R}^d} \left| \int \mu(x) \mu(\theta x + z) dx \right|^2 dz d\theta.$$

By Plancherel in x ,

$$= \int_{O(d)} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \widehat{\mu}(\xi) e^{2\pi i z \cdot \xi} \widehat{\mu}(\theta \xi) d\xi \right|^2 dz d\theta.$$

By Plancherel in z ,

$$\begin{aligned} &= \int_{O(d)} \left(\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\widehat{\mu}(\theta \xi)|^2 d\xi \right) d\theta \\ &= \int \left(\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\omega \right)^2 r^{d-1} dr. \end{aligned}$$

A new derivation of the Mattila integral

Suppose $E_1, \dots, E_{k+1} \subset \mathbb{R}^d$ with Borel measures μ_j on E_j . Suppose

$$\Phi(x^1, x^2, \dots, x^{k+1}) = \Phi(y^1, y^2, \dots, y^{k+1})$$

if and only if for some $g \in G$, a LCT group,

$$(y^1, y^2, \dots, y^{k+1}) = (gx^1, gx^2, \dots, gx^{k+1}).$$

Let $\mu^\epsilon(x) = \mu * \phi_\epsilon(x) \in C_0^\infty$, where $\phi \in C_0^\infty$, $\int \phi = 1$, $\phi_\epsilon(\cdot) = \epsilon^{-d} \phi(\frac{\cdot}{\epsilon})$.

Theorem (B.L., 2017)

$$\int_G \prod_{j=1}^{k+1} \left(\int_{\mathbb{R}^d} \mu_j^\epsilon(x) \mu_j^\epsilon(gx) dx \right) dg < \infty$$

uniformly in ϵ implies that the set

$$\Delta_\Phi(E_1, \dots, E_{k+1}) := \{\Phi(x^1, \dots, x^{k+1}) : x^j \in E_j\}$$

has positive Lebesgue measure.

Sum-product problems

Given any $A \subset \mathbb{R}$ of N points, one can define its sum set, product set by

- $A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$;
- $AA = \{a_1 a_2 : a_1, a_2 \in A\}$.

The Erdős-Szemerédi conjecture states that for any $\epsilon > 0$,

$$\max\{\#(A + A), \#(AA)\} \gtrsim_{\epsilon} N^{2-\epsilon}.$$

Two extremal cases are

- $A = \{1, 2, 3, \dots, N\}$;
- $A = \{1, 2, 4, \dots, 2^N\}$.

Roughly speaking, this conjecture reflects that a set of real numbers cannot be structured in both additive and multiplicative sense simultaneously. The best currently known result in \mathbb{R} is due to Konyagin and Shkredov. They prove

$$\max\{\#(A + A), \#(AA)\} \gtrsim_{\epsilon} N^{\frac{4}{3} + \frac{5}{9813} - \epsilon}.$$

Sum-product problems

There are various formulations of sum-product estimates. For example, Balog conjectured that

$$\#(A(A + A)) \gtrsim_{\epsilon} N^{2-\epsilon}$$

and the best currently known result is given by Murphy, Roche-Newton and Shkredov. They prove

$$\#(A(A + A)) \gtrsim_{\epsilon} N^{\frac{3}{2} + \frac{5}{242} - \epsilon}.$$

One can also consider sum-product estimates on $A \subset \mathbb{F}_p$, a finite field, where p is a prime. It was first studied by Bourgain, Katz, Tao and later studied by different authors.

Continuous sum-product problem

Discrete version	Continuous version
Erdős distance conjecture $\#(\Delta(P)) \geq C_\epsilon N^{\frac{2}{d}-\epsilon}$	Falconer distance conjecture $ \Delta(E) > 0$ if $\dim_{\mathcal{H}}(E) > \frac{d}{2}$
Sum-product conjectures $\max\{\#(A+A), \#(AA)\} \gtrsim_\epsilon N^{2-\epsilon}$ $\#(A(A+A)) \gtrsim_\epsilon N^{2-\epsilon}$?

Counterexamples show that for any $s \in [0, 1]$, there exists $A, B \in [0, 1]$, $\dim_{\mathcal{H}}(A) = \dim_{\mathcal{H}}(A+A) = s$, $\dim_{\mathcal{H}}(B) = \dim_{\mathcal{H}}(BB) = s$.

Theorem (B.L., 2017)

Suppose $E, F, H \subset \mathbb{R}^2$, $\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) + \dim_{\mathcal{H}}(H) > 4$. Then

$$E \cdot (F + H) := \{x \cdot (y + z) : x \in E, y \in F, z \in H\}$$

has positive Lebesgue measure. It is sharp, with $E = \{0\}$, $F = H = [0, 1]^2$.

Continuous sum-product problem

Taking $E = A \times \{0\}$, $F = B \times [0, 1]$, $H = C \times [0, 1]$, we obtain the following sum-product estimate on \mathbb{R} .

Corollary (B.L., 2017)

Suppose $A, B, C \subset \mathbb{R}$ and $\dim_{\mathcal{H}}(A) + \dim_{\mathcal{H}}(B) + \dim_{\mathcal{H}}(C) > 2$, then

$$|A(B + C)| > 0.$$

This dimensional exponent is generally optimal. In particular, for any $A \subset \mathbb{R}$,

$$|A(A + A)| > 0$$

whenever $\dim_{\mathcal{H}}(A) > \frac{2}{3}$.

To see the sharpness, one can take $A = \{0\}$, $B = C = [0, 1]$.

Sketch of the proof

For any $x = (x_1, x_2) \in \mathbb{R}^2$, denote $x^\perp = (x_2, -x_1)$ and $E^\perp = \{x^\perp : x \in E\}$. Without loss of generality, we may work on

$$E \cdot (F + H)^\perp.$$

Notice $x \cdot y^\perp = x' \cdot y'^\perp$ if and only if there exists $g \in SL_2(\mathbb{R})$ such that $x' = gx$, $y' = gy$. Therefore the corresponding Mattila integral is

$$\int_{SL_2(\mathbb{R})} \left(\int_{\mathbb{R}^d} \mu_1^\epsilon(x) \mu_1^\epsilon(gx) dx \right) \left(\int_{\mathbb{R}^d} \mu_2^\epsilon(x) \mu_2^\epsilon(gx) dx \right) dg,$$

where $\mu_1 = \mu_E$, $\mu_2 = \mu_F * \mu_H$. By Plancherel, it suffices to show

$$\int \left(\int_{\mathbb{R}^d} \widehat{\mu}_E(\xi) \overline{\widehat{\mu}_E(g\xi)} d\xi \right) \left(\int_{\mathbb{R}^d} \widehat{\mu}_F(\eta) \widehat{\mu}_H(\eta) \overline{\widehat{\mu}_F(g\eta) \widehat{\mu}_H(g\eta)} d\eta \right) dg < \infty.$$

We may take μ_E, μ_F, μ_H as Frostman measures and it follows from harmonic analysis techniques.

Continuous sum-product problem

Although the theorem and corollary above are generally sharp, one can still expect a better dimensional exponent for $E \cdot (E + E)$, $E \subset \mathbb{R}^2$ and $A(A + A)$, $A \subset \mathbb{R}$. For instance, it is not hard to show

Theorem

Suppose $E, F, H \subset \mathbb{R}^2$ and $\dim_{\mathcal{H}}(E) > 1$, $\dim_{\mathcal{H}}(F) + \dim_{\mathcal{H}}(H) > 2$, then

$$|E \cdot (F - H)| > 0.$$

In particular,

$$|E \cdot (E \pm E)| > 0$$

if $\dim_{\mathcal{H}}(E) > 1$, where the dimensional threshold is optimal.

Sketch of the proof

The proof is super simple if we only consider $E \cdot (E - E)$. By Marstrand projection theorem, for almost all $e \in S^1$,

$$|\pi_e(A)| > 0 \quad (1)$$

and

$$|\{t \in \mathbb{R} : \dim_{\mathcal{H}}(E \cap \pi_e^{-1}(t)) \geq \dim_{\mathcal{H}}(E) - 1\}| > 0. \quad (2)$$

Then one can choose two distinct points $x, y \in E$ such that $x - y$ is parallel to e where $|\pi_e(A)| > 0$. Hence

$$|E \cdot (E - E)| \geq |E \cdot (x - y)| = |\pi_e(E)| > 0.$$

For general $E \cdot (F - H)$, we need to apply a radial projection theorem due to Mattila and Orponen to show $F - H$ contains a lot of directions. Then there must exist one direction e in $F - H$ such that

$$|E \cdot (F - H)| \geq |\pi_e(E)| > 0$$

Continuous sum-product problem

Corollary

$$|A(A + A) + A(A + A)| > 0$$

whenever $A \subset \mathbb{R}$, $\dim_{\mathcal{H}}(A) > \frac{1}{2}$. This dimensional exponent is sharp.

Now what we know is, for $A \subset \mathbb{R}$,

- $|A(A + A)| > 0$ if $\dim_{\mathcal{H}}(A) > \frac{2}{3}$;
- $|A(A + A) + A(A + A)| > 0$ if $\dim_{\mathcal{H}}(A) > \frac{1}{2}$, which is sharp.

Then it is reasonable to make the following conjecture as an analog of Balog's conjecture.

Conjecture

Suppose $A \subset \mathbb{R}$, $\dim_{\mathcal{H}}(A) > \frac{1}{2}$, then $A(A + A)$ has positive Lebesgue measure.

Counterexamples

Let $\{q_i\}$ be a positive sequence such that $q_{i+1} \geq q_i^i$. Take $A_i \subset \mathbb{R}$ as the $q_i^{-\frac{1}{s}}$ neighborhood of

$$[0, 1] \cap \frac{1}{q_i} \mathbb{Z}$$

and denote

$$A = \bigcap_{i=1}^{\infty} A_i.$$

It is known that $\dim_{\mathcal{H}}(A) = s$ while for each $i = 1, 2, \dots$, $A(A + A)$, $(A + A)(A + A)$, etc. are contained in the $q_i^{-\frac{1}{s}}$ neighborhood of

$$[0, 1] \cap \frac{1}{q_i^2} \mathbb{Z}$$

whose Lebesgue measure is $q_i^{2-\frac{1}{s}} \rightarrow 0$ if $s < \frac{1}{2}$.

In fact, instead of a constant s , we can choose $s_i \nearrow \frac{1}{2}$ such that

$q_i^{2-\frac{1}{s_i}} \rightarrow 0$. Then $\dim_{\mathcal{H}}(A) = \frac{1}{2}$ while $|A(A + A)| = \dots = 0$.

Thank You!