Nature is an infinite sphere of which the center is everywhere and the circumference nowhere.

Blaise Pascal 1623-1662

p-adic Cantor strings and complex fractal dimensions

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Cantor set

Henry John Stephen Smith discovered the Cantor set in 1874. Georg Ferdinand Ludwig Philipp Cantor introduced the Cantor set as an example of a perfect set that is nowhere dense in the real line $\mathbb{R}$ in 1883.

- Ternary Cantor set

$$\mathcal{C} = \left\{ c \in [0,1] : c = a_0 + \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots, a_i \in \{0,2\} \text{for all } i \geq 0 \right\}$$

- $\mathcal{C}$ is self-similar:

$$\mathcal{C} = \varphi_1(\mathcal{C}) \cup \varphi_2(\mathcal{C})$$

where $\varphi_1(x) = \frac{x}{3}$ and $\varphi_1(x) = \frac{x}{3} + \frac{2}{3}$.

Cantor fractal string

Michel Lapidus considered the complement of the Cantor set in [0,1] as an infinite sequence of lengths in 1991.

- The ordinary Cantor string $\mathcal{CS} = \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ with the corresponding multiplicity $1, 2, 4, \ldots$.
- Let $s \in \mathbb{C}$ and consider the geometric zeta function associated with the Cantor string

$$\zeta_{\mathcal{CS}}(s) = \frac{1}{3^s} + \frac{2}{3^{2s}} + \frac{4}{3^{3s}} + \cdots = \frac{1}{3^s} \sum_{n=0}^{\infty} \left(\frac{2}{3^s}\right)^n = \frac{1}{3^s - 2}$$

- Complex dimensions are poles of the geometric zeta function. They are $\omega = \frac{\log 2}{\log 3} + in\frac{2\pi}{\log 3} = D + inp, n \in \mathbb{Z}$, for the Cantor string $\mathcal{CS}$.

Complex fractal dimensions

- $$\sigma_{\mathcal{CS}} = \frac{\log 2}{\log 3} = D_M$$

  where $D_M = \inf\{\alpha \geq 0 : V_{\mathcal{CS}}(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\}$ is the Minkowski dimension of the Cantor string and $\sigma_{\mathcal{CS}} = \inf\{\alpha \in \mathbb{R} : \sum_{n=1}^{\infty} m_n \cdot l_n^{\alpha} < \infty\}$ is the abscissa of convergence of the Dirichlet series defining the geometric zeta function $\zeta_{\mathcal{CS}}$.

Theorem (M. L. Lapidus)

Let $\mathcal{L}$ be a real fractal string with infinitely many nonzero lengths, then $\sigma_{\mathcal{L}} = D_M$.

- Complex dimensions reveal oscillations intrinsic to the geometry, spectrum and dynamic of the fractal string.

Kurt Hensel field of *p*-adic numbers

- $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ wrt the *p*-adic norm $|\cdot|_p : \mathbb{Q} \longrightarrow [0, \infty)$ given by $|x|_p = p^{-v}$ and $|0|_p = 0$.
- $(\mathbb{Q}_p, |\cdot|_p)$ is an ultrametric space since $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.
- $(\mathbb{Q}_p, |\cdot|_p)$ is a nonarchimedean field since $|x + x|_p \leq |x|_p$.
- $(\mathbb{Q}_p, |\cdot|_p)$ is locally compact and totally disconnected.
- $(\mathbb{R}, |\cdot|) = (\mathbb{Q}_\infty, |\cdot|_\infty)$ is the archimedean field at infinity.
- The topological boundary of a *p*-adic ball is empty and every point in the *p*-adic ball is a center!

Alexander Ostrowski

Theorem (Ostrowski Theorem)

Every completion of $\mathbb{Q}$ is equivalent to $\mathbb{Q}_p$ for some prime $p \leq \infty$.

- $\mathbb{Q}_p = \{a_v p^v + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots \mid v \in \mathbb{Z}, a_i \in \{0, 1, 2, 3, \ldots, p-1\}\}$
- The unit ball in $\mathbb{Q}_p$ is the ring of p-adic integers $\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots | a_i \in \{0, 1, 2, 3, \ldots, p-1\}\}$

Nonarchimedean 3-adic Cantor set and string

- The 3-adic Cantor set $\mathcal{C}_3$ is the self-similar set generated by the family of similarity contraction mappings $\{\phi_1(x) = 3x, \phi_2(x) = 3x + 2\}$ of $\mathbb{Z}_3$ into itself.
- $\mathcal{C}_3 = \{x \in \mathbb{Z}_3 \mid x = a_0 + a_1 3 + a_2 3^3 + \cdots, a_i \in \{0, 2\} \text{ for all } i \geq 0\}$
- $\mathcal{C}_3$ is naturally homeomorphic to the ternary Cantor set $\mathcal{C}$.
- The 3-adic Cantor string $\mathcal{CS}_3$ is the complement of $\mathcal{C}_3$ in $\mathbb{Z}_3$.
- $\mathcal{CS}_3 = (1 + 3\mathbb{Z}_3) \cup (3 + 9\mathbb{Z}_3) \cup (5 + 9\mathbb{Z}_3) \cup \cdots$ is isometric to the archimedean Cantor string $\mathcal{CS}$.
- Complex dimensions of $\mathcal{CS}_3$ are $\omega = \frac{\log 2}{\log 3} + in\frac{2\pi}{\log 3}$

5-adic Cantor set and string by Chugh, Kumar & Rani

- The 5-adic Cantor set $\mathcal{C}_5$ is the self-similar set generated by the family of similarity contraction mappings $\{\phi_1(x) = 5x, \phi_2(x) = 5x + 2, \phi_3(x) = 5x + 4\}$.
- $\mathcal{C}_5 = \{x \in \mathbb{Z}_5 \mid x = a_0 + a_1 5 + a_2 5^2 + \cdots, a_i \in \{0, 2, 4\}$ for all $i \geq 0\}$
- The nonarchimedean 5-adic Cantor set $\mathcal{C}_5$ is homeomorphic to the archimedean quinary Cantor set $\mathcal{C}_5^*$.
- $\mathcal{CS}_5 = (1 + 5\mathbb{Z}_5) \cup (3 + 5\mathbb{Z}_5) \cup (5 + 25\mathbb{Z}_5) \cup (15 + 25\mathbb{Z}_5) \cup \cdots$ is isometric to the archimedean quinary Cantor string $\mathcal{CS}_5^*$.
- Complex dimensions of the 5-adic Cantor string $\mathcal{CS}_5$ are $\omega = \frac{\log 3}{\log 5} + in\frac{2\pi}{\log 5}, n \in \mathbb{Z}$.

p-adic Cantor sets and strings

- For $p > 2$, the p-adic Cantor set $\mathcal{C}_p$ is the self-similar set generated by the family of similarity contraction mappings $\{\phi_1(x) = px, \phi_2(x) = px + 2, \ldots, \phi_{\frac{p+1}{2}}(x) = px + p - 1\}$
- $\mathcal{C}_p = \{x \in \mathbb{Z}_p \mid x = a_0 + a_1 p + a_2 p^2 + \cdots, a_i \in \{0, 2, \ldots, p-1\}$ for all $i \geq 0\}$
- The nonarchimedean p-adic Cantor set $\mathcal{C}_p$ is homeomorphic to the archimedean pinary Cantor set $\mathcal{C}_p^*$
- The p-adic Cantor string $\mathcal{CS}_p$ is the complement of the p-adic Cantor set $\mathcal{C}_p$ in $\mathbb{Z}_p$
- The geometric zeta function of the p-adic Cantor string is $\zeta_{\mathcal{CS}_p}(s) = \frac{p-1}{2p^s - p - 1}$
- Complex dimensions of $\mathcal{CS}_p$ are $\omega = \frac{\log \frac{p+1}{2}}{\log p} + in\frac{2\pi}{\log p}$ and the residue of $\zeta_{\mathcal{CS}_p}$ at ω is $\frac{p-1}{(p+1)\log p}$

Complex fractal dimensions

Theorem

Let $\mathcal{L}_p$ be a p-adic fractal strings with infinitely many nonzero lengths, then $\sigma_{\mathcal{L}_p} = D_M$.

Complex dimensions reveal oscillations in the geometry of p-adic fractal strings

Exact tube formula for self-similar strings

$$V_{\mathcal{CS}_p}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{CS}_p}} \frac{res(\zeta_{\mathcal{CS}_p}; \omega)}{p(1-\omega)} \varepsilon^{1-\omega}$$

$$V_{\mathcal{CS}_p}(\varepsilon) = \sum_{n \in \mathbb{Z}} \frac{p-1}{p(p+1)\log p} \frac{\varepsilon^{1-D-\frac{2\pi i n}{\log p}}}{(1-D-\frac{2\pi i n}{\log p})}$$

$$V_{\mathcal{CS}_p}(\varepsilon) = \frac{p-1}{p(p+1)\log p} \varepsilon^{1-D} \sum_{n \in \mathbb{Z}} \frac{\cos(\frac{2\pi n}{\log p}\log \varepsilon) - i \sin(\frac{2\pi n}{\log p}\log \varepsilon)}{1-D-\frac{2\pi i n}{\log p}}$$

$\Re(\omega) = D$ represents the amplitude of the logarithmic oscillations in the geometry of the fractal string and $\Im(\omega) = \frac{2\pi n}{\log p}$ represents the frequency.

p-adic self-similar strings are not Minkowski measurable

p-adic Cantor strings $\mathcal{CS}_p$ are not Minkowski measurable:

$$\lim_{\varepsilon \to 0^+} \frac{V_{\mathcal{CS}_p}(\varepsilon)}{\varepsilon^{1-D_M}}$$

doesn't exist in $(0, \infty)$

Theorem

All p-adic self-similar strings are lattice and lattice strings are never Minkowski measurable.

Average Minkowski content

Average Minkowski content is the logarithmic Cesàro average:

$$\mathcal{M}_{av}(\mathcal{L}_p) = \lim_{T\to\infty} \frac{1}{\log T} \int_{1/T}^{1} \frac{V_{\mathcal{L}_p}(\varepsilon)}{\varepsilon^{1-D}} \frac{d\varepsilon}{\varepsilon}$$

Theorem

Let $\mathcal{L}_p$ be a p-adic self-similar string of dimension D, then

$$\mathcal{M}_{av}(\mathcal{L}_p) = \frac{res(\zeta_{\mathcal{L}_p}; D)}{p(1-D)}$$

$$\mathcal{M}_{av}(\mathcal{CS}_p) = \frac{1}{p(1-\frac{\log\frac{p+1}{2}}{\log p})} \frac{p-1}{(p+1)\log p}$$

Adelic Cantor strings and global complex dimensions

$$\mathcal{CS} \times \mathcal{CS}_2 \times \mathcal{CS}_3 \times \mathcal{CS}_5 \times \mathcal{CS}_7 \times \cdots \subset \mathbb{A}_{\mathbb{Z}} = \mathbb{R} \times \prod_{p<\infty} \mathbb{Z}_p$$

$$(\mathcal{CS}_2 \times \mathcal{CS}_2^*) \times (\mathcal{CS}_3 \times \mathcal{CS}_3^*) \times (\mathcal{CS}_5 \times \mathcal{CS}_5^*) \times (\mathcal{CS}_7 \times \mathcal{CS}_7^*) \cdots$$

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Thαnk you for listening