The viscous Burgers Equation on the Sierpinski gasket

Melissa Meinert

Bielefeld University

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Burgers Equation (Burgers '48)

Viscous Burgers Equation (BE):

$$\frac{\partial u}{\partial t} + \underbrace{(u \cdot \nabla)\, u}_{\textit{convection}} = \nu \Delta u, \qquad \nu > 0$$

(BE) describes laminar flow in fluid dynamics.

Aim:

1. Formulation on SG
2. Existence of solutions

For related results (existence, uniqueness and regularity of the solution for (BE)) we refer to Liu and Qian [LQ].

Starting point

Consider the Cauchy problem for the Heat Equation (HE):

$$\begin{cases} w_t(x,t) = \nu \Delta w(x,t), & t > 0 \\ w(x,0) = w_0(x) \end{cases}$$

with $\inf_{x \in SG} w_0(x) > c_0 > 0$, $w_0 \in C(SG)$.

Idea: Use knowledge about (HE) and Cole Hopf Transformation [Col51, Hop50]

$$u(x,t) := -2\nu \frac{(w(x,t))_x}{w(x,t)}$$

to proof existence of solutions!

Setup

- $X = SG$ Sierpinski gasket
- μ finite Borel measure s.t. $\mu(U) > 0 \ \forall U \subset SG$ open, $U \neq \emptyset$
- $(\mathcal{E}, \mathcal{F})$ standard resistence form
- Δ_μ Laplacian, defined by

$$\mathcal{E}(u, v) = -\int v \Delta_\mu u d\mu$$

for all $v \in \mathcal{F}$ vanishing on the boundary

See Kusuoka [Kus89], Kigami [Kig89, Kig93, Kig01] and Strichartz [Str06].

Remark

resistance form + SG compact in resistance
$\Rightarrow \mathcal{F} \subset C(SG)$ only contains bounded functions
$\mathcal{F}$ *algebra*, see [BH91, Cor. I.3.3.2], and it holds

$$\mathcal{E}(fg)^{\frac{1}{2}} \leq \|f\|_\infty \mathcal{E}(g)^{\frac{1}{2}} + \|g\|_\infty \mathcal{E}(f)^{\frac{1}{2}} \qquad \forall f, g \in \mathcal{F}$$

Following [HRT13], see also [IRT12], we use the framework of 1-forms and derivations introduced by Cipriani and Sauvageot [CS03].

$$\mathcal{F} \otimes \mathcal{F} := \left\{ \sum_i f_i \otimes g_i \text{ finite linear combination, } f_i, g_i \in \mathcal{F} \right\}$$

Definition ([CS03])

On $\mathcal{F} \otimes \mathcal{F}$ let

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} = \frac{1}{2} \left(\mathcal{E}(a, cdb) + \mathcal{E}(abd, c) - \mathcal{E}(bd, ac) \right)$$

and $\| \cdot \|_{\mathcal{H}}$ the associated semi-norm. The space $\mathcal{H}$ is obtained by taking the quotient by the normzero subspace and then the completion in $\| \cdot \|_{\mathcal{H}}$.

$$\begin{aligned} a\,(b \otimes c) &= (ab) \otimes c - a \otimes (bc) \\ (b \otimes c)\, d &= b \otimes cd \qquad\qquad a, b, c, d \in \mathcal{F} \end{aligned}$$

Theorem

[IRT12, Thm. 5.6] If $\mathcal{H}$ is the Hilbert module of a resistance form on a finitely ramified cell structure, then $\bigcup_{n=0}^{\infty} \mathcal{H}_n$ is dense in $\mathcal{H}$, with the subspaces

$$\mathcal{H}_n = \left\{ \sum_{w \in W_n} h_w \otimes \mathbb{1}_{K_w} \right\},$$

where the sum is over all n-cells, $\mathbb{1}_{K_w}$ is the indicator of the n-cell K_w, and h_w is a n-harmonic function modulo additive constants. Further, if P is the projection from $\mathcal{H}$ onto the closure of the image of the derivation ∂, then

$$\bigcup_{n=0}^{\infty} \left\{ \sum_{w \in W_n} h_w \otimes \mathbb{1}_{K_w} = f \otimes \mathbb{1}, \ \ \textit{where } f \textit{ is n-harmonic} \right\}$$

is dense in $P\mathcal{H}$.

Definition (abstract derivation)

A *derivation operator* $\partial : \mathcal{F} \to \mathcal{H}$ can be defined by setting

$$\partial f := f \otimes \mathbb{1}, \qquad f \in \mathcal{F}.$$

It obeys the *Leibniz property*

$$\partial(ab) = (\partial a)b + a(\partial b).$$

and is a bounded linear operator satisfying

$$\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f), \qquad f \in \mathcal{F}.$$

Remark

The operator $\partial : \mathcal{F} \to \mathcal{H}$ can be extended to a closed linear operator $\partial_\mu : L^2(SG, \mu) \to \mathcal{H}$ with domain dense in $\mathcal{F}$. For $f \in \mathcal{F}$ and $F \in C^1(\mathbb{R})$ the *chain rule* is also satisfied

$$\partial F(f) = F'(f)\partial f.$$

In our case:

$$\mathcal{E}(f) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{p\in V_m} \sum_{q\sim^m p} (f(p) - f(q))^2$$

with domain $\mathcal{F}$

$$\begin{aligned}\|g\partial f\|_{\mathcal{H}}^2 &= \|f \otimes g\|_{\mathcal{H}}^2 \\ &= \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{p\in V_m} \sum_{q\sim^m p} \left(\frac{g(p)+g(q)}{2}\right)^2 (f(p) - f(q))^2\end{aligned}$$

Definition (divergence)

The *divergence* $\partial_\mu^* : \mathcal{H} \to L^2(SG, \mu)$ is defined as -adjoint operator to ∂_μ, equipped with the domain

$$\mathcal{D}(\partial_\mu^*) := \left\{ v \in \mathcal{H} : \exists u \in L^2(SG, \mu) : \langle u, \phi \rangle_{L^2(SG,\mu)} = -\langle v, \phi \rangle_{\mathcal{H}} \, \forall \phi \in \mathcal{F} \right\}.$$

For $v \in \mathcal{D}(\partial_\mu^*)$ set $\partial_\mu^* v := u$.

Remark

For $f \in \mathcal{D}(\Delta_\mu)$ the following is true:

$$\partial_\mu f \in \mathcal{D}(\partial_\mu^*) \qquad \text{and} \qquad \Delta_\mu f = \partial_\mu^* \partial_\mu f.$$

We will consider $f \in \mathcal{D}(\Delta_\mu)$ such that

$$\Delta_\mu f \in C(SG) \subset L^2(SG, \mu) \tag{1}$$

Definition

We denote with

$$\mathcal{D}_{\mathcal{H}\to C(SG)}(\partial_\mu^*) := \left\{v \in \mathcal{D}(\partial_\mu^*) : \partial_\mu^* v \in C(SG)\right\}$$

the space of test vector fields.
Further, for $u \in \mathcal{H}$, $v \in \mathcal{D}_{\mathcal{H}\to C(SG)}(\partial_\mu^*)$ we define

$$\begin{aligned} \left(\partial_\mu \partial_\mu^* u\right)(v) &:= -(\partial_\mu^* u)(\partial_\mu^* v), \\ \partial_\mu \langle u, u\rangle_{\mathcal{H}} &:= -\langle (\partial_\mu^* v) u, u\rangle_{\mathcal{H}}. \end{aligned}$$

For f as in (1), we have $\partial f \in \mathcal{D}_{\mathcal{H}\to C(SG)}(\partial_\mu^*)$.

Existence of weak solution

Definition

Let $u_0 \in \mathcal{H}$. We say that a function $u : [0, \infty) \to \mathcal{H}$ with initial condition u_0 is a *weak solution of the abstract Burgers Equation*, if the function is differentiable on $(0, \infty)$ and obeys for all $v \in \mathcal{D}_{\mathcal{H} \to C_b}(\partial_\mu^*)$

$$\begin{cases} \Delta_\mu u(v) - \partial_\mu \langle u(t), u(t) \rangle_{\mathcal{H}}(v) &= \langle u_t(t), v \rangle_{\mathcal{H}}, \qquad t > 0 \\ \lim_{t \to 0} \langle u(t) - u_0, v \rangle_{\mathcal{H}} &= 0. \end{cases} \tag{2}$$

Existence of weak solution

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Theorem

Let $w_0 \in C_b$ a positively function with $w_0(p) \geq c_0$, $p \in SG$, for a fixed constant $c_0 > 0$. Then the function $u : [0, \infty) \to \mathcal{H}$,

$$u(t) := -\partial_\mu(\log w(t)), \qquad t > 0, \qquad \textit{with } u_0 = -\partial_\mu \log w_0$$

is a weak solution of the initial problem (2).

References I

[BH91] Nicolas Bouleau and Francis Hirsch.
Dirichlet forms and analysis on Wiener space, volume 14 of *De Gruyter studies in mathematics ; 14*.
de Gruyter, Berlin [u.a.], 1991.

[Col51] Julian D Cole.
On a quasi-linear parabolic equation occurring in aerodynamics.
Quarterly of applied mathematics, 9(3):225–236, 1951.

[CS03] Fabio Cipriani and Jean-Luc Sauvageot.
Derivations as square roots of dirichlet forms.
JOURNAL OF FUNCTIONAL ANALYSIS, 2003, 201, 1, 78, 2003.

[Hop50] Eberhard Hopf.
The partial differential equation u sub t + uu sub x = mu sub xx.
DTIC AND NTIS, 1950.

[HRT13] Michael Hinz, Michael Rockner, and Alexander Teplyaev.
Vector analysis for dirichlet forms and quasilinear pde and spde on metric measure spaces.
STOCHASTIC PROCESSES AND THEIR APPLICATIONS, 2013, 123, 12, 4373, 2013.

[IRT12] Marius Ionescu, Luke G. Rogers, and Alexander Teplyaev.
Derivations and dirichlet forms on fractals.
JOURNAL OF FUNCTIONAL ANALYSIS, 2012, 263, 8, 2141, 2012.

References II

[Kig89] Jun Kigami.
A harmonic calculus on the sierpinski spaces.
Japan Journal of Applied Mathematics, 6(2):259–290, 1989.

[Kig93] Jun Kigami.
Harmonic calculus on p.c.f. self-similar sets.
Transactions of the American Mathematical Society, 335(2):721–755, 1993.

[Kig01] Jun Kigami.
Analysis on fractals, volume 143 of *Cambridge tracts in mathematics ; 143*.
Cambridge Univ. Press, Cambridge, 2001.

[Kus89] Shigeo Kusuoka.
Dirichlet forms on fractals and products of random matrices.
Kyoto University. Research Institute for Mathematical Sciences. Publications, 1989, 25, 4, 659, 1989.

[LQ] Xuan Liu and Zhongmin Qian.
Brownian motion on the sierpinski gasket and related stochastic differential equations.
Submitted, arXiv:1612.01297v2.

[Str06] Robert S. Strichartz.
Differential equations on fractals.
Princeton Univ. Press, Princeton [u.a.], 2006.