The Homotopy Critical Spectrum for Non-Geodesic Spaces

Conrad Plaut

Fractals 6, Cornell University

June 13, 2017

Outline

1. Discrete Homotopy Theory

Outline

1. Discrete Homotopy Theory
2. Applications for Geodesic Spaces

Outline

1. Discrete Homotopy Theory
2. Applications for Geodesic Spaces
3. Issues for Non-geodesic Metric Spaces

Outline

1. Discrete Homotopy Theory
2. Applications for Geodesic Spaces
3. Issues for Non-geodesic Metric Spaces
4. Chained Metric Spaces

Outline

1. Discrete Homotopy Theory
2. Applications for Geodesic Spaces
3. Issues for Non-geodesic Metric Spaces
4. Chained Metric Spaces
5. Elaboration will be for informal discussion of resistance metrics

A little history

- 2001 Berestovskii-P.: generalized covering spaces of topological groups based on a construction of Schreier from the 1920's, rediscovered by Malcev in the 1940's, reinterpreted by us in terms of discrete chains and homotopies

A little history

- 2001 Berestovskii-P.: generalized covering spaces of topological groups based on a construction of Schreier from the 1920's, rediscovered by Malcev in the 1940's, reinterpreted by us in terms of discrete chains and homotopies
- 2001: Sormani-Wei independently developed the idea of δ-covers of geodesics spaces, using a construction of Spanier

A little history

- 2001 Berestovskii-P.: generalized covering spaces of topological groups based on a construction of Schreier from the 1920's, rediscovered by Malcev in the 1940's, reinterpreted by us in terms of discrete chains and homotopies
- 2001: Sormani-Wei independently developed the idea of δ-covers of geodesics spaces, using a construction of Spanier
- 2007 Berestovskii-P. extended discrete homotopy ideas to uniform spaces (hence metric spaces)

A little history

- 2001 Berestovskii-P.: generalized covering spaces of topological groups based on a construction of Schreier from the 1920's, rediscovered by Malcev in the 1940's, reinterpreted by us in terms of discrete chains and homotopies
- 2001: Sormani-Wei independently developed the idea of δ-covers of geodesics spaces, using a construction of Spanier
- 2007 Berestovskii-P. extended discrete homotopy ideas to uniform spaces (hence metric spaces)
- 2013 P.-Wilkins focused on metric spaces, showed that Berestovskii-P. and Sormani-Wei constructions are essentially equivalent for geodesic spaces, proved some fundamental group finiteness theorems

A little history

- 2001 Berestovskii-P.: generalized covering spaces of topological groups based on a construction of Schreier from the 1920's, rediscovered by Malcev in the 1940's, reinterpreted by us in terms of discrete chains and homotopies
- 2001: Sormani-Wei independently developed the idea of δ-covers of geodesics spaces, using a construction of Spanier
- 2007 Berestovskii-P. extended discrete homotopy ideas to uniform spaces (hence metric spaces)
- 2013 P.-Wilkins focused on metric spaces, showed that Berestovskii-P. and Sormani-Wei constructions are essentially equivalent for geodesic spaces, proved some fundamental group finiteness theorems
- 2015 Jim Conant, Victoria Curnutte, Corey Jones, P., Kristen Pueschel, Maria Lusby, Wilkins: Bad things can happen with non-geodesic spaces

Discrete Homotopies in a Metric Space

Let X be a metric space.

Definition

For $\varepsilon > 0$, an ε-chain is a finite sequence $\{x_0, \ldots, x_n\}$ such that for all i, $d(x_i, x_{i+1}) < \varepsilon$.

Discrete Homotopies in a Metric Space

Let X be a metric space.

Definition

For $\varepsilon > 0$, an ε-chain is a finite sequence $\{x_0, \ldots, x_n\}$ such that for all i, $d(x_i, x_{i+1}) < \varepsilon$.

Definition

An ε-homotopy consists of a finite sequence $\langle \gamma_0, \ldots, \gamma_n \rangle$ of ε-chains, where each γ_i differs from its predecessor by a "basic move": adding or removing a *single* point, always leaving the endpoints fixed.

Epsilon-Covers

Definition

Fixing a basepoint $*$, X_ε is defined to be the set of all ε-homotopy equivalence classes of ε-chains starting at $*$, and $\phi_\varepsilon : X_\varepsilon \longrightarrow X$ is the endpoint map. Equivalence classes are denoted by $[\alpha]_\varepsilon$.

Epsilon-Covers

Definition

Fixing a basepoint $*$, X_ε is defined to be the set of all ε-homotopy equivalence classes of ε-chains starting at $*$, and $\phi_\varepsilon : X_\varepsilon \longrightarrow X$ is the endpoint map. Equivalence classes are denoted by $[\alpha]_\varepsilon$.

Definition

The group $\pi_\varepsilon(X)$ is the subset of X_ε consisting of classes of ε-loops starting and ending at $*$ with operation induced by concatenation, i.e., $[\alpha]_\varepsilon * [\beta]_\varepsilon = [\alpha * \beta]_\varepsilon$.

The “Lifted Metric”

- There is a natural metric on X_ε with the following properties

The “Lifted Metric”

- There is a natural metric on X_ε with the following properties
- When X is connected, ϕ_ε is a covering map with deck group $\pi_\varepsilon(X)$ (acting by preconcatenation)

The "Lifted Metric"

- There is a natural metric on X_ε with the following properties
- When X is connected, ϕ_ε is a covering map with deck group $\pi_\varepsilon(X)$ (acting by preconcatenation)
- $\pi_\varepsilon(X)$ acts as isometries on X_ε

The "Lifted Metric"

- There is a natural metric on X_ε with the following properties
- When X is connected, ϕ_ε is a covering map with deck group $\pi_\varepsilon(X)$ (acting by preconcatenation)
- $\pi_\varepsilon(X)$ acts as isometries on X_ε
- $\phi_\varepsilon : X_\varepsilon \to X$ is an isometry from any $\frac{\varepsilon}{2}$-ball onto its image

Homotopy Critical Values

Definition

An ε-loop λ in a metric space X is called ε-critical if λ is not ε-null, but is δ-null for all $\delta > \varepsilon$. When an ε-critical ε-loop exists, ε is called a homotopy critical value; the collection of these values is called the Homotopy Critical Spectrum.

Homotopy Critical Values

Definition

An ε-loop λ in a metric space X is called ε-critical if λ is not ε-null, but is δ-null for all $\delta > \varepsilon$. When an ε-critical ε-loop exists, ε is called a homotopy critical value; the collection of these values is called the Homotopy Critical Spectrum.

- For geodesic spaces, this spectrum is discrete in $(0, \infty)$ and determines when the equivalence type of the covering spaces changes

Homotopy Critical Values

Definition

An ε-loop λ in a metric space X is called ε-critical if λ is not ε-null, but is δ-null for all $\delta > \varepsilon$. When an ε-critical ε-loop exists, ε is called a homotopy critical value; the collection of these values is called the Homotopy Critical Spectrum.

- For geodesic spaces, this spectrum is discrete in $(0, \infty)$ and determines when the equivalence type of the covering spaces changes
- Homotopy critical values are determined by lengths of "essential circles", which are very special closed geodesics.

Homotopy Critical Values

Definition

An ε-loop λ in a metric space X is called ε-critical if λ is not ε-null, but is δ-null for all $\delta > \varepsilon$. When an ε-critical ε-loop exists, ε is called a homotopy critical value; the collection of these values is called the Homotopy Critical Spectrum.

- For geodesic spaces, this spectrum is discrete in $(0, \infty)$ and determines when the equivalence type of the covering spaces changes
- Homotopy critical values are determined by lengths of "essential circles", which are very special closed geodesics.
- Therefore the homotopy critical spectrum corresponds to a subset of the length spectrum and differs from the Sormani-Wei "covering spectrum" by a factor of $\frac{2}{3}$.

Laplace vs Length vs HCS/CS spectra

- The Laplace and Length Spectra are related, but the relationship is not fully understood

Laplace vs Length vs HCS/CS spectra

- The Laplace and Length Spectra are related, but the relationship is not fully understood
- For example, any two flat tori are isospectral (same Laplace Spectrum) if and only if they have the same length spectrum (ignoring multiplicity)

Laplace vs Length vs HCS/CS spectra

- The Laplace and Length Spectra are related, but the relationship is not fully understood
- For example, any two flat tori are isospectral (same Laplace Spectrum) if and only if they have the same length spectrum (ignoring multiplicity)
- In general it is not known whether the length spectrum is a spectral invariant, i.e. whether spaces with the same Laplace spectrum must have the same length spectrum

Laplace vs Length vs HCS/CS spectra

- The Laplace and Length Spectra are related, but the relationship is not fully understood
- For example, any two flat tori are isospectral (same Laplace Spectrum) if and only if they have the same length spectrum (ignoring multiplicity)
- In general it is not known whether the length spectrum is a spectral invariant, i.e. whether spaces with the same Laplace spectrum must have the same length spectrum
- de Smit, Gornet, and Sutton showed that the Covering Spectrum (hence HCS) is not a spectral invariant

Laplace vs Length vs HCS/CS spectra

- The Laplace and Length Spectra are related, but the relationship is not fully understood
- For example, any two flat tori are isospectral (same Laplace Spectrum) if and only if they have the same length spectrum (ignoring multiplicity)
- In general it is not known whether the length spectrum is a spectral invariant, i.e. whether spaces with the same Laplace spectrum must have the same length spectrum
- de Smit, Gornet, and Sutton showed that the Covering Spectrum (hence HCS) is not a spectral invariant
- That is, there exist isospectral manifolds with different covering spectra

Generalizations of Gromov, Anderson, Shen-Wei Theorems

Theorem

(P.-Wilkins) Suppose X is a semilocally simply connected, compact geodesic space of diameter D, and let $\varepsilon > 0$. Then for any choice of basepoint, $\pi_1(X)$ has a set of generators $g_1, \ldots, g_k$ of length at most $2D$ and relations of the form $g_i g_m = g_j$ with

$$k \leq \frac{8(D+\varepsilon)}{\varepsilon} \cdot \Gamma(X,\varepsilon) \cdot C\left(X, \frac{\varepsilon}{4}\right)^{\frac{8(D+\varepsilon)}{\varepsilon}} .$$

Generalizations of Gromov, Anderson, Shen-Wei Theorems

Theorem

(P.-Wilkins) Suppose X is a semilocally simply connected, compact geodesic space of diameter D, and let $\varepsilon > 0$. Then for any choice of basepoint, $\pi_1(X)$ has a set of generators $g_1, ..., g_k$ of length at most $2D$ and relations of the form $g_i g_m = g_j$ with

$$k \leq \frac{8(D+\varepsilon)}{\varepsilon} \cdot \Gamma(X,\varepsilon) \cdot C\left(X, \frac{\varepsilon}{4}\right)^{\frac{8(D+\varepsilon)}{\varepsilon}}.$$

Corollary

Let $\mathcal{X}$ be any Gromov-Hausdorff precompact class of semilocally simply connected compact geodesic spaces. If there are numbers $\varepsilon > 0$ and N such that for every $X \in \mathcal{X}$, $\Gamma(X,\varepsilon) \leq N$, then there are finitely many possible fundamental groups for spaces in $\mathcal{X}$.

Examples: The Good, The Bad, and The Ugly

- The geodesic circle of length L has exactly one homotopy critical value: $\frac{L}{3}$

Examples: The Good, The Bad, and The Ugly

- The geodesic circle of length L has exactly one homotopy critical value: $\frac{L}{3}$
- In general, a compact, semilocally simply connected geodesic space "unrolls" step by step to its universal cover

Examples: The Good, The Bad, and The Ugly

- The geodesic circle of length L has exactly one homotopy critical value: $\frac{L}{3}$
- In general, a compact, semilocally simply connected geodesic space "unrolls" step by step to its universal cover
- A circle with a "gap" in it has a "phantom" ε-cover even though it is simply connected

Examples: The Good, The Bad, and The Ugly

- The geodesic circle of length L has exactly one homotopy critical value: $\frac{L}{3}$
- In general, a compact, semilocally simply connected geodesic space "unrolls" step by step to its universal cover
- A circle with a "gap" in it has a "phantom" ε-cover even though it is simply connected
- The students mentioned earlier worked on a project showing that HCS of "topologist's combs" have cluster points away from 0

Examples: The Good, The Bad, and The Ugly

- The geodesic circle of length L has exactly one homotopy critical value: $\frac{L}{3}$
- In general, a compact, semilocally simply connected geodesic space "unrolls" step by step to its universal cover
- A circle with a "gap" in it has a "phantom" ε-cover even though it is simply connected
- The students mentioned earlier worked on a project showing that HCS of "topologist's combs" have cluster points away from 0
- Jay Wilkins showed that there are metric spaces whose homotopy critical spectrum is $[0, 1]$.

Refinement is the Key

- In a geodesic space, when $0 < \delta < \varepsilon$ then any ε-chain can be "refined" to a δ-chain in the same ε-homotopy class

Refinement is the Key

- In a geodesic space, when $0 < \delta < \varepsilon$ then any ε-chain can be "refined" to a δ-chain in the same ε-homotopy class
- This is done by joining x_i and x_{i+1} by a geodesic and subdividing the geodesics into δ-chains

Refinement is the Key

- In a geodesic space, when $0 < \delta < \varepsilon$ then any ε-chain can be "refined" to a δ-chain in the same ε-homotopy class
- This is done by joining x_i and x_{i+1} by a geodesic and subdividing the geodesics into δ-chains
- Refinement is critical to statements like "close ε-chains are ε-homotopic"

Refinement is the Key

- In a geodesic space, when $0 < \delta < \varepsilon$ then any ε-chain can be "refined" to a δ-chain in the same ε-homotopy class
- This is done by joining x_i and x_{i+1} by a geodesic and subdividing the geodesics into δ-chains
- Refinement is critical to statements like "close ε-chains are ε-homotopic"
- These concepts fail for the "bad" examples above

Chained Metric Spaces

Definition

A metric space X is called "chained" if whenever $d(x, y) < \varepsilon$ and $0 < \delta < \varepsilon$, then x and y can be joined by a δ-chain that lies entirely in $B(x, \varepsilon) \cap B(y, \varepsilon)$. Equivalently, $B(x, \varepsilon) \cap B(y, \varepsilon)$ is "chain connected".

- Geodesic spaces are chained since any geodesic joining x and y lies in $B(x, \varepsilon) \cap B(y, \varepsilon)$.

Chained Metric Spaces

Definition

A metric space X is called "chained" if whenever $d(x, y) < \varepsilon$ and $0 < \delta < \varepsilon$, then x and y can be joined by a δ-chain that lies entirely in $B(x, \varepsilon) \cap B(y, \varepsilon)$. Equivalently, $B(x, \varepsilon) \cap B(y, \varepsilon)$ is "chain connected".

- Geodesic spaces are chained since any geodesic joining x and y lies in $B(x, \varepsilon) \cap B(y, \varepsilon)$.
- If (X, d) is a chained metric space and $f : [0, \infty) \to [0, \infty)$ is a concave increasing function such that $f(0) = 0$ then $(X, f \circ d)$ is chained.

Chained Metric Spaces

Definition

A metric space X is called "chained" if whenever $d(x, y) < \varepsilon$ and $0 < \delta < \varepsilon$, then x and y can be joined by a δ-chain that lies entirely in $B(x, \varepsilon) \cap B(y, \varepsilon)$. Equivalently, $B(x, \varepsilon) \cap B(y, \varepsilon)$ is "chain connected".

- Geodesic spaces are chained since any geodesic joining x and y lies in $B(x, \varepsilon) \cap B(y, \varepsilon)$.
- If (X, d) is a chained metric space and $f : [0, \infty) \to [0, \infty)$ is a concave increasing function such that $f(0) = 0$ then $(X, f \circ d)$ is chained.
- So for example, if (X, d) is a geodesic metric then $(X, d^{\frac{1}{2}})$ has no rectifiable curves but is still chained.

Chained Metric Spaces

Definition

A metric space X is called "chained" if whenever $d(x, y) < \varepsilon$ and $0 < \delta < \varepsilon$, then x and y can be joined by a δ-chain that lies entirely in $B(x, \varepsilon) \cap B(y, \varepsilon)$. Equivalently, $B(x, \varepsilon) \cap B(y, \varepsilon)$ is "chain connected".

- Geodesic spaces are chained since any geodesic joining x and y lies in $B(x, \varepsilon) \cap B(y, \varepsilon)$.
- If (X, d) is a chained metric space and $f : [0, \infty) \to [0, \infty)$ is a concave increasing function such that $f(0) = 0$ then $(X, f \circ d)$ is chained.
- So for example, if (X, d) is a geodesic metric then $(X, d^{\frac{1}{2}})$ has no rectifiable curves but is still chained.
- A stronger (and more geometrically appealing) condition is: Every $x, y \in X$ may be joined by a curve $c : [0, 1] \to X$ such that $d(x, c(t))$ is increasing and $d(y, (c(t))$ is decreasing.

Finiteness

Theorem

Let X be a compact chained metric space and $\varepsilon > 0$. Then there are at most

$$2^{C(X,\frac{\varepsilon}{4})^{40C(X,\frac{\varepsilon}{2})}}$$

homotopy critical values δ such that $\delta \geq \varepsilon$. In particular, the homotopy critical spectrum is discrete in $(0,\infty)$. Moreover, this number is uniformly bounded in any Gromov-Hausdorff precompact class.

Thank You