Riemann-Liouville Fractional Calculus of Coalescence Hidden-variable Fractal Interpolation Functions

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6^{th} Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals
June 13-17, 2017

Outline

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Fractal Interpolation Function (FIF)

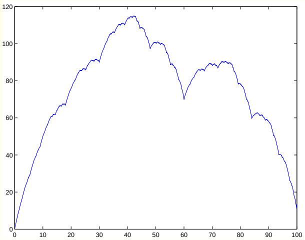
- Fractal Interpolation Function (FIF) : [Barnsley M.F., 1986]

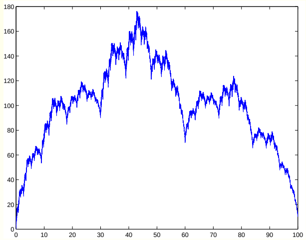
- **Similarities of FIF and traditional methods**
  - ∗ Geometrical Character - can be plotted on graph
  - ∗ Represented by formulas

- **Difference between FIF and traditional methods**
  - ∗ Fractal Character

Coalescence Hidden-variable Interpolation Functions

- For simulating curves that exhibit self-affine and non-self-affine nature simultaneously, Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) was introduced by [Chand A.K.B. and Kapoor G.P., 2007].





Construction of a CHFIF

- Given data$\{(x_k, y_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$
- Generalized data $\{(x_k, y_k, z_k) \in \mathbb{R}^3 : k = 0, 1, \ldots, N\}$
- $[x_0, x_N] = I,\ [x_{k-1}, x_k] = I_k,\ k = 1, 2, \ldots, N$
- $L_k : I \to I_k$

$$\begin{aligned} L_k(x_0) &= a_k x + b_k \\ &= \frac{x_k - x_{k-1}}{x_N - x_0}(x - x_0) + x_{k-1} \end{aligned} \tag{1}$$

Construction of a CHFIF

- $F_k : I \times \mathbb{R}^2 \to \mathbb{R}^2$

$$F_k(x, y, z) = \big(\alpha_k y + \beta_k z + p_k(x), \gamma_k z + q_k(x)\big) \qquad (2)$$

- $|\alpha_k| < 1$, $|\gamma_k| < 1$, $|\beta_k| + |\gamma_k| < 1$

-

$$F_k(x_0, y_0, z_0) = (y_{k-1}, z_{k-1})$$
$$F_k(x_N, y_N, Z_N) = (y_k, z_k)$$

- $\omega_k : I \times \mathbb{R}^2 \to I \times \mathbb{R}^2$

$$\omega_k(x, y, z) = (L_k(x), F_k(x, y, z)),\ k = 1, 2, \dots N$$

Construction of a CHFIF

Theorem ([Chand A.K.B. and Kapoor G.P., 2007])

(1) $\{I \times \mathbb{R}^2; \omega_k, k = 1, 2, \ldots, N\}$ *is a hyperbolic IFS with respect to a metric equivalent to Euclidean metric on* $\mathbb{R}^3$*.*

(2) The attractor $G \subseteq \mathbb{R}^3$ *such that* $G = \bigcup_{k=1}^{N} \omega_k(G)$ *of the above IFS is graph of a continuous function* $f : I \to \mathbb{R}^2$ *such that* $f(x_k) = (y_k, z_k)$ *for* $k = 0, 1, \ldots, N$ *i.e.* $G = \{(x, f(x)) : x \in I \text{ and } f(x) = (y(x), z(x))\}$*.*

Definition

The **Coalescence Hidden-variable Fractal Interpolation Function (CHFIF)** for the given interpolation data $\{(x_k, y_k) : k = 0, 1, \dots, N\}$ is defined as the continuous function $f_1 : I \to \mathbb{R}$, where f_1 is the first component of the continuous function $f = (f_1, f_2)$, graph of which is attractor of the hyperbolic IFS.

- f_2 - AFIF (Self-Affine Fractal Interpolation Function)
- $y_k = z_k$ and $\alpha_k + \beta_k = \gamma_k$ for all k, $f_1 = f_2$ is FIF

Construction of CHFIF

- CHFIF : if $x_{k-1} \leq x \leq x_k$ then

$$f_1(x) = \alpha_k f_1(L_k^{-1}(x)) + \beta_k f_2(L_k^{-1}(x)) + p_k(L_k^{-1}(x))$$

- FIF : if $x_{k-1} \leq x \leq x_k$ then

$$f_2(x) = \gamma_k f_2(L_k^{-1}(x)) + q_k(L_k^{-1}(x))$$

Outline

Riemann-Liouville fractional integral

Definition

Let $-\infty < a < x < b < \infty$. The Riemann-Liouville fractional integral of order $\nu > 0$ with lower limit a is defined for locally integrable functions $f : [a, b] \to \mathbb{R}$ as

$$I_{a+}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt \quad (3)$$

for $x > a$.

- Given data$\{(x_k, y_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$
- Generalized data $\{(x_k, y_k, z_k) \in \mathbb{R}^3 : k = 0, 1, \ldots, N\}$

$$p_k^\nu(x) = a_k^\nu \; I_{x_0+}^\nu \, p_k(x) + \frac{1}{\Gamma(\nu)} \int_{x_0}^{x_{k-1}} (L_k(x) - t)^{\nu-1} \, f_1(t) \, dt \tag{4}$$

and

$$q_k^\nu(x) = a_k^\nu \; I_{x_0+}^\nu \, q_k(x) + \frac{1}{\Gamma(\nu)} \int_{x_0}^{x_{k-1}} (L_k(x) - t)^{\nu-1} \, f_2(t) dt. \tag{5}$$

Riemann-Liouville fractional integral

$$\begin{aligned} F_k^\nu(x,y,z) &= \left(F_{k,1}^\nu(x,y,z), F_{k,2}^\nu(x,z)\right) \\ &= \left(a_k^\nu \alpha_k y + a_k^\nu \beta_k z + p_k^\nu(x), a_k^\nu \gamma_k z + q_k^\nu(x)\right) \end{aligned} \tag{6}$$

Define

$$\omega_k^\nu(x,y,z) = \left(L_k(x), F_k^\nu(x,y,z)\right); \tag{7}$$

$y_0^\nu = 0 = z_0^\nu$,

$$\begin{aligned} z_N^\nu &= \frac{q_N^\nu(x_N)}{1 - a_N^\nu \gamma_N}, \\ y_N^\nu &= \frac{a_N^\nu \beta_N}{1 - a_N^\nu \alpha_N}\, z_N^\nu + \frac{p_N^\nu(x_N)}{1 - a_N^\nu \alpha_N}, \\ z_k^\nu &= a_k^\nu \gamma_k z_N^\nu + q_k^\nu(x_N) = q_{k+1}^\nu(x_0) \\ \text{and} \quad y_k^\nu &= a_k^\nu \alpha_k y_N^\nu + a_k^\nu \beta_k z_N^\nu + p_k^\nu(x_N) = p_{k+1}^\nu(x_0),\ k = 1, 2, \ldots, N-1. \end{aligned} \tag{8}$$

Riemann-Liouville fractional integral of FIF

Proposition

Let f_2 *be a FIF passing through the interpolation data given by* $\{(x_k, z_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$*. Then, Riemann-Liouville fractional integral of a FIF of order* ν *is also a FIF passing through the data* $\{(x_k, z_k^\nu) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$*, where* z_k^ν *are given by* (8).

Theorem ([S.A.P, 2017])

Let f_1 be the CHFIF passing through the interpolation data given by $\{(x_k, y_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$ and f_2 be the corresponding FIF passing through the data $\{(x_k, z_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$. Then, Riemann-Liouville fractional integral of a CHFIF of order ν given by (3) *is also a CHFIF passing through the data $\{(x_k, y_k^\nu) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$, where y_k^ν are given by* (8).

Riemann-Liouville fractional integral of CHFIF

Sketch of Proof:

Let x such that $x_{k-1} < x < x_k$ for some $k \in \{1, 2, \dots, N\}$. Then,

$$
\begin{aligned}
I^{\nu}_{x_0+} f_1(x) &= \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} f_1(t)dt \\
&= \frac{1}{\Gamma(\nu)} \left\{ \int_{x_0}^{x_{k-1}} (x-t)^{\nu-1} f_1(t)\, dt + \int_{x_{k-1}}^{x} (x-t)^{\nu-1} f_1(t)\, dt \right\}
\end{aligned}
$$

Riemann-Liouville fractional integral of CHFIF

$$
\begin{aligned}
I^{\nu}_{x_0+} f_1(x) &= \frac{1}{\Gamma(\nu)} \left\{ \int_{x_0}^{x_{k-1}} (x-t)^{\nu-1} f_1(t)\, dt \right. \\
&\qquad \left. + a_k^{\nu} \int_{x_0}^{L_k^{-1}(x)} (L_k^{-1}(x)-t)^{\nu-1} f_1(L_k(t))\, dt \right\} \\
&= a_k^{\nu}\, \alpha_k\, I^{\nu}_{x_0+} f_1(L_k^{-1}(x)) + a_k^{\nu}\, \beta_k\, I^{\nu}_{x_0+} f_2(L_k^{-1}(x)) \\
&\qquad + a_k^{\nu}\, I^{\nu}_{x_0+} p_k(L_k^{-1}(x)) + \frac{1}{\Gamma(\nu)} \left\{ \int_{x_0}^{x_{k-1}} (x-t)^{\nu-1} f_1(t)\, dt \right\}
\end{aligned}
$$

Outline

Riemann-Liouville fractional derivative

Definition

Let $-\infty < a < x < b < \infty$, $0 < \nu$, $f \in L_1([a,b])$ and $I^{n-\nu}f \in W^{1,1}$, where n is the smallest integer greater than ν . The Riemann-Liouville fractional derivative of order ν with lower limit a is defined as

$$(D^{\nu}_{a+}f)(x) = \frac{d^n}{dx^n}(I^{n-\nu}_{a+}\, f)(x)$$

and $(D^{\nu}_{a+}\, f)(x) = f(x)$ when $\nu = 0$.

Riemann-Liouville fractional derivative of FIF

$$q_k^{d\nu}(x) = a_k^{-\nu} D^{\nu} q_k(x) + \frac{a_k^{-n}}{\Gamma(n-\nu)} \frac{d^n}{dx^n} \left[\int_{x_0}^{x_{k-1}} f_2(t)(L_k(x)-t)^{n-\nu-1}\, dt \right] \quad (9)$$

and

$$p_k^{d\nu}(x) = a_k^{-\nu} D^{\nu}\, p_k(x) + \frac{a_k^{-n}}{\Gamma(n-\nu)} \frac{d^n}{dx^n} \left[\int_{x_0}^{x_{k-1}} f_1(t)(L_k(x)-t)^{n-\nu-1}\, dt \right]. \quad (10)$$

Riemann-Liouville fractional derivative of FIF

Proposition

Let f_2 be a FIF passing through the interpolation data $\{(x_k, z_k) \in \mathbb{R}^2 : k = 0, 1, \dots, N\}$ and $|\gamma_k| < a_k^{\nu}$ for some fixed $\nu > 0$. Then Riemann-Liouville fractional derivative of a FIF of order ν is also a FIF provided (9) *is satisfied.*

Theorem ([S.A.P, 2017])

Let f_1 be the CHFIF passing through the interpolation data given by $\{(x_k, y_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$ and f_2 be the corresponding FIF passing through the data $\{(x_k, z_k) \in \mathbb{R}^2 : k = 0, 1, \ldots, N\}$. For a fixed $\nu > 0$, if the free variables and constrained variables are such that $|\alpha_k| < a_k^\nu$, $|\gamma_k| < a_k^\nu$ and $|\beta_k| + |\gamma_k| < a_k^\nu$ then Riemann-Liouville fractional derivative of a CHFIF of order ν is also a CHFIF provided (9) *and* (10) *are satisfied.*

Riemann-Liouville fractional derivative of FIF

Suppose f_2 is a FIF passing through interpolation data given by $\{(x_k, z_k) : k = 0, 1, 2, \ldots, N\}$ constructed with the free variables γ_k for $k = 1, 2, \ldots, N$. Then, for all ν satisfying

$$\nu < \frac{\log |\gamma_k|}{\log a_k}$$

Riemann-Liouville fractional derivative of f_2 of order ν exists and is a FIF provided (9) is satisfied.

Riemann-Liouville fractional derivative of CHFIF

Suppose f_1 is a CHFIF passing through a interpolation data given by $\{(x_k, y_k) : k = 0, 1, 2, \ldots, N\}$ constructed with the free variables α_k, γ_k and constrained variables β_k for $k = 1, 2, \ldots, N$. Then, for all ν satisfying

$$\nu < \min\left\{\frac{\log|\alpha_k|}{\log a_k}, \frac{\log(|\beta_k| + |\gamma_k|)}{\log a_k}\right\}$$

Riemann-Liouville fractional derivative of f_1 of order ν exists and is a CHFIF provided (9) and (10) are satisfied.

Example

Blancmange Curve:[Takagi, 1903]

$$B(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n} \quad x \in [0,1],$$

where, $s(y) = \min_{m \in \mathbb{Z}} |y - m|, y \in \mathbb{R}$.

$B\left(\frac{x+k-1}{2}\right) = \frac{1}{2}B(x) + \frac{k-1+(-1)^{k-1}x}{2} \quad x \in [0,1]$ for $k = 1, 2$.

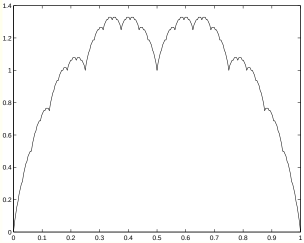


Figure: **Blanchmange Curve**

Example

- $\gamma_k = \frac{1}{2}$ for $k = 1, 2$.
- $L_k(x) = \frac{1}{2}x + \frac{k-1}{2}$
- $q_k(x) = \frac{k-1+(-1)^{k-1}x}{2}$
- $\nu < \frac{\log|\gamma_k|}{\log a_k} = \frac{\log 1/2}{\log 1/2} = 1$
- $q_1^{d\nu}(x) = \frac{1}{2^{1-\nu}\,\Gamma(2-\nu)}x^{1-\nu}$

Example

$$
\begin{aligned}
q_2^{d\nu}(x) = {} & \frac{1}{2^{1-\nu}\,\Gamma(1-\nu)}\left(x^{\nu} - \frac{x^{1-\nu}}{(1-\nu)}\right) \\
& - \frac{1}{\Gamma(1-\nu)}\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{m=1}^{2^n}(-1)^{m-1}\times \\
& \times\left\{\left[2^n\frac{\left(\frac{x+1}{2}-\frac{m}{2^{n+1}}\right)^{1-\nu}-\left(\frac{x+1}{2}-\frac{m-1}{2^{n+1}}\right)^{1-\nu}}{(1-\nu)}\right]\right. \\
& +\left(\frac{x+1}{2}-\frac{m}{2^{n+1}}\right)^{-\nu}\left(\frac{m}{2}-A\right) \\
& \left.-\left(\frac{x+1}{2}-\frac{m-1}{2^{n+1}}\right)^{-\nu}\left(\frac{m-1}{2}-A\right)\right\}
\end{aligned}
$$

$$
A = \begin{cases} m/2 & \text{if m is even} \\ (m-1)/2 & \text{if m is odd} \end{cases}
$$

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Thank You!