Log-Minkowski measurability and complex dimensions

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Relative fractal drum (A, Ω)

- $\emptyset \neq A \subset \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, Lebesgue measurable, i.e., $|\Omega| < \infty$
- δ-neighbourhood of A:

$$A_\delta = \{x \in \mathbb{R}^N : d(x, A) < \delta\}$$

- **upper** r**-dimensional Minkowski content of** (A, Ω):

$$\overline{\mathcal{M}}^r(A, \Omega) := \limsup_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}$$

- **upper Minkowski dimension of** (A, Ω):

$$\overline{\dim}_B(A, \Omega) = \inf\{r \in \mathbb{R} \ : \ \overline{\mathcal{M}}^r(A, \Omega) = 0\}$$

- **lower Minkowski content** and **dimension** defined via $\liminf$

Minkowski measurability

- $\underline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega) \Rightarrow \exists \dim_B(A,\Omega)$
- if $\exists D \in \mathbb{R}$ such that

$$0 < \underline{\mathcal{M}}^D(A,\Omega) = \overline{\mathcal{M}}^D(A,\Omega) < \infty,$$

  we say (A,Ω) is **Minkowski measurable**; in that case $D = \dim_B(A,\Omega)$
- if the above inequalities are not satisfied for D, we call (A,Ω) **Minkowski degenerated**

The relative distance zeta function

- (A, Ω) RFD in $\mathbb{R}^N$, $s \in \mathbb{C}$ and **fix** $\delta > 0$
- the **distance zeta function of** (A, Ω):

$$\zeta_{A,\Omega}(s; \delta) := \int_{A_\delta \cap \Omega} d(x, A)^{s-N}\, dx$$

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- the **complex dimensions of** (A, Ω) are defined as the poles of $\zeta_{A,\Omega}$
- take Ω to be an open neighborhood of A in order to recover the classical ζ_A

Holomorphicity theorem for the relative distance zeta function [LapRaŽu]

Theorem

- (A, Ω) RFD in $\mathbb{R}^N$:

(*a*) $\zeta_{A,\Omega}(s)$ is **holomorphic on** $\{\mathrm{Re}\, s > \overline{\dim}_B(A,\Omega)\}$

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(c) if $\exists D = \dim_B(A, \Omega) < N$ and $\underline{\mathcal{M}}^D(A, \Omega) > 0$, then
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- we call $\{\operatorname{Re} s = \overline{\dim}_B(A,\Omega)\}$ the **critical line**

(Generalized) complex dimensions of an RFD

Definition

Let W be a connected open set s.t. $\{\operatorname{Re} s > \overline{\dim}_B(A,\Omega)\} \subset W$ and $\zeta_{A,\Omega}$ is holomorphic on W. The set of **visible complex dimensions of** (A,Ω) (**with respect to** W) is the set of singularities $\mathcal{P}(\zeta_{A,\Omega}, W) \subset \partial W$ of $\zeta_{A,\Omega}$.

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principal complex dimensions:

$$\dim_{PC}(A,\Omega) := \{\omega \in \mathcal{P}(\zeta_{A,\Omega}, W) : \mathrm{Re}\, \omega = \overline{\dim}_B(A,\Omega)\}. \quad (1)$$

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- includes poles, essential and nonisolated singularities (accumulation of poles, natural boundaries)
- branching points (W is then a subset of the appropriate Riemann surface) and also “mixed singularities”

Fractal tube formulas for relative fractal drums

- An asymptotic formula for the **tube function** $t \mapsto |A_t \cap \Omega|$ as $t \to 0^+$ in terms of $\zeta_{A,\Omega}$.

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Theorem (Simplified pointwise formula with error term)

• $\alpha < \overline{\dim}_B(A,\Omega) < N$; $\zeta_{A,\Omega}$ satisfies suitable rational growth conditions (d-**languidity**) on the half-plane $\mathbf{W} := \{\mathrm{Re}\, s > \alpha\}$, then:

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega}, \mathbf{W})} \mathrm{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s), \omega\right) + O(t^{N-\alpha}).$$

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- if we allow polynomial growth of $\zeta_{A,\Omega}$, in general, we get a tube formula in the sense of Schwartz distributions

The Minkowski measurability criterion

Theorem (Minkowski measurability criterion)

- (A, Ω) is such that $\exists D := \dim_B(A, \Omega)$ and $D < N$
- $\zeta_{A,\Omega}$ is d-languid on a suitable domain $W \supset \{\operatorname{Re} s = D\}$

Then, the following is equivalent:

(a) (A, Ω) is Minkowski measurable.

(b) D is the only pole of $\zeta_{A,\Omega}$ located on the critical line $\{\operatorname{Re} s = D\}$ and it is simple.

$$\mathcal{M}^D(A, \Omega) = \frac{\operatorname{res}(\zeta_{A,\Omega}, D)}{N - D}$$

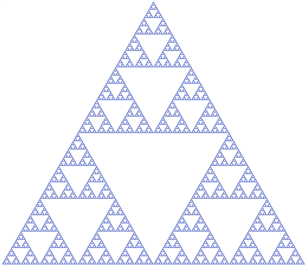
The Minkowski measurability criterion

- $(a) \Rightarrow (b)$: from the distributional tube formula and the **Uniqueness theorem for almost periodic distributions** due to **Schwartz**

- $(b) \Rightarrow (a)$: a consequence of a **Tauberian theorem** due to **Wiener** and **Pitt** (conditions can be considerably weakened)

- the assumption $D < N$ can be removed by appropriately embedding the RFD in $\mathbb{R}^{N+1}$

Figure: The Sierpiński gasket



- an example of a **self-similar fractal spray** with a generator G being an open equilateral triangle and with **scaling ratios** $r_1 = r_2 = r_3 = 1/2$
- $(A, \Omega) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{3} (r_j A, r_j \Omega)$

Fractal tube formula for The Sierpiński gasket

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}$$

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$$|A_t| = \sum_{\omega\in\mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta),\omega\right)$$

$$= t^{2-\log_2 3}\frac{6\sqrt{3}}{\log 2}\sum_{k=-\infty}^{+\infty}\frac{(4\sqrt{3})^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k-1)\omega_k} + \left(\frac{3\sqrt{3}}{2}+\pi\right)t^2,$$

valid pointwise for all $t \in (0, 1/2\sqrt{3})$.

Gauge Minkowski content [HeLap]

If (A,Ω) is Minkowski degenerate, $\exists D := \dim_B(A,\Omega)$ and

$$|A_t \cap \Omega| = t^{N-D}(F(t) + o(1)) \quad \text{as } t \to 0^+, \tag{2}$$

where $F(t) = h(t)$ or $F(t) = 1/h(t)$ for $h : (0,\varepsilon_0) \to (0,+\infty)$, $h(t) \to +\infty$ as $t \to 0^+$ and $h \in O(t^\beta)$ for $\forall \beta < 0$.

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- h is called a **gauge function of slow growth to** $+\infty$ **at** 0^+
- $1/h$ is called a **gauge function of slow decay** 0 **at** 0^+
- typical gauge functions: $\left(\log^{\circ k} t^{-1}\right)^a$ for $a \in \mathbb{R}^*, k \in \mathbb{N}$

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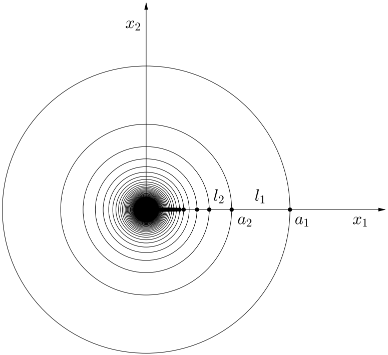
- $1/h$ is called a **gauge function of slow decay 0 at 0^+**
- typical gauge functions: $\left(\log^{\circ k} t^{-1}\right)^a$ for $a \in \mathbb{R}^*, k \in \mathbb{N}$
- *h*-**Minkowski content**: $\mathcal{M}^D(A, \Omega, h) = \lim_{t \to 0^+} \frac{|A_t \cap \Omega|}{t^{N-D} h(t)}$.

The fractal nest generated by the a-string



$a > 0$, $a_j := j^{-a}$, $\ell_j := j^{-a} - (j+1)^{-a}$, $\Omega := B_{a_1}(0)$

$$\zeta_{A_a,\Omega}(s) = \frac{2^{2-s}\pi}{s-1} \sum_{j=1}^{\infty} \ell_j^{s-1}(a_j + a_{j+1})$$

Fractal tube formula for the fractal nest generated by the a-string

Example

$$\mathcal{P}(\zeta_{A_a,\Omega}) \subseteq \left\{1, \frac{2}{a+1}, \frac{1}{a+1}\right\} \cup \left\{-\frac{m}{a+1} : m \in \mathbb{N}\right\}$$

$a \neq 1,\ D := \frac{2}{1+a} \Rightarrow$

$$|(A_a)_t \cap \Omega| = \frac{2^{2-D} D\pi}{(2-D)(D-1)} a^{D-1} t^{2-D} + 2\pi\big(2\zeta(a) - 1\big) t + O\big(t^{2-\frac{1}{a+1}}\big), \text{ as } t \to 0^+$$

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$$|(A_1)_t \cap \Omega| = \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_{A_1,\Omega}(s), 1\right) + o(t)$$
$$= 2\pi t(-\log t) + \text{const} \cdot t + o(t) \quad \text{as } t \to 0^+$$

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- a pole ω of order m generates terms of type $t^{N-\omega}(-\log t)^{k-1}$ for $k = 1, \dots, m$ in the fractal tube formula

Sufficiency for log-Minkowski measurability via the Wiener-Pitt Tauberian theorem

- $m \in \mathbb{Z}$; $\zeta_{A,\Omega}^{[m]}$ denotes its the $|m|$-th derivative if $m < 0$ and the m-th primitive if $m > 0$;

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Theorem

- $\overline{D} := \overline{\dim}_B(A, \Omega) < N$; $\exists m \in \mathbb{Z}$, $\exists K > 0$, *s.t.* $\forall \lambda > 0$

$$G_x(y) := \zeta_{A,\Omega}^{[m]}(x + \mathbb{i}y) - \frac{(-1)^m K}{x + \mathbb{i}y - \overline{D}}$$

converges in $L^1(-\lambda, \lambda)$ *to a boundary function* $G(y)$ *as* $x \to \overline{D}^+$.

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converges in $L^1(-\lambda, \lambda)$ *to a boundary function* $G(y)$ *as* $x \to \overline{D}^+$. *Then,* $\exists D := \dim_B(A,\Omega) = \overline{D}$ *and* (A,Ω) *is h-Minkowski measurable s.t.*

$$\mathcal{M}^D(A,\Omega,h) = \frac{K}{N-D}, \tag{3}$$

where $h(t) := (-\log t)^m$.

Corollary: Case of poles

Theorem

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Then, $\exists D := \dim_B(A,\Omega) = \overline{D}$ *and* (A,Ω) *is h-Minkowski measurable:*

$$\mathcal{M}^D(A,\Omega,h) = \frac{\zeta_{A,\Omega}[D]_{-m}}{(N-D)(m-1)!}, \tag{4}$$

where $h(t) := (-\log t)^{m-1}$.

- $\zeta_{A,\Omega}[D]_{-m}$ denotes the leading coefficient of the Laurent expansion of $\zeta_{A,\Omega}$ at D.

Zero-log singularities

Definition

- ψ, ϕ holo. germs at $\omega \in \mathbb{C}$ s.t. ω is a zero of order m of ψ.

We say that the holo. germ

$$f(s) := \psi(s)\,\mathrm{Log}(s-\omega) + \phi(s)$$

on the principal branch of $\mathrm{Log}(s-\omega)$ has a **zero-log singularity of order** m at ω.

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• for instance, $f(s) = (s-2)^3\,\mathrm{Log}(s-2)$ has a zero-log singularity of order 3 at $\omega = 2$

• $\mathrm{Log}\, s$ has a zero-log singularity of order 0 at $\omega = 0$, etc.

Corollary: Case of zero-log singularities

Theorem (Case of zero-log singularities)

- $\overline{D} := \overline{\dim}_B(A, \Omega) < N$; $\dim_{PC}(A, \Omega)$ *consists only of zero-log singularities and has no accumulation points;*
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$$\mathcal{M}^D(A, \Omega, h) = (-1)^{m+1} m! \lim_{s \to D} \frac{\psi(s)}{(s - D)^m}, \tag{5}$$

where $h(t) := \dfrac{1}{(-\log t)^{m+1}}$.

Examples of sets that have zero-log complex dimensions?

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Let $f \in \mathrm{Diff}^r(0,a)$ *be continuous on* $[0,a)$ *, positive on* $(0,a)$ *and let* $f(0) = f'(0) = 0$. *Assume* $1 < x(\log(f))'(x)$. *Put* $g = id - f$ *and let* $S^g(x_0) = \{x_n | n \in \mathbb{N}\}$ *be an orbit of* g, $x_0 < a$.

$$|A_t(S^g(x_0))| \asymp g^{-1}(t)$$

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- Let $g(x) = x^k \log^{\circ m}(x^{-1})$, then $g^{-1}(t) \asymp \dfrac{t^{1/k}}{\log^{\circ m}(t^{-1})^{1/k}}$
- we also have $|A_t(S^g(x_0))| \asymp t(-\log t)$ for appropriate differentiable f

The $1/2$-square fractal

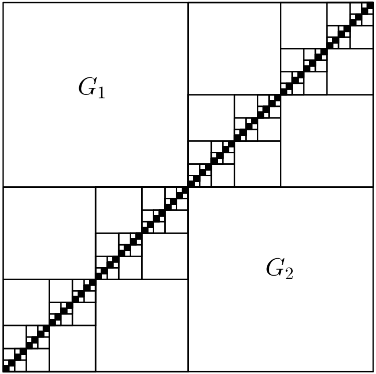

Figure: Here, $G := G_1 \cup G_2$ is the single generator of the corresponding self-similar spray or RFD (A, Ω), where $\Omega = (0,1)^2$.

Fractal tube formula for the $1/2$-square fractal

$$\zeta_A(s) = \frac{2^{-s}}{s(s-1)(2^s-2)} + \frac{4}{s-1} + \frac{2\pi}{s}, \tag{6}$$

$$D(\zeta_A) = 1, \quad \mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \{0\} \cup \left(1 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right). \tag{7}$$

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$$\begin{aligned} |A_t| &= \sum_{\omega \in \mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s), \omega\right) \\ &= \frac{1}{4\log 2} t \log t^{-1} + t\, G\left(\log_2 (4t)^{-1}\right) + \frac{1+2\pi}{2} t^2, \end{aligned} \tag{8}$$

valid for all $t \in (0, 1/2)$, where G is a nonconstant 1-periodic function on $\mathbb{R}$ bounded away from zero and ∞.
The $1/2$-square fractal is **critically fractal** in dimension 1.

Further research directions

- Riemann surfaces generated by relative fractal drums
- Extending the notion of complex dimensions to include complicated “mixed” singularities/branching points and connecting them with various gauge functions
- Obtaining corresponding tube formulas and gauge-Minkowski measurability criteria
- Applying the theory to problems from dynamical systems

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