A criterion for box-counting measurability

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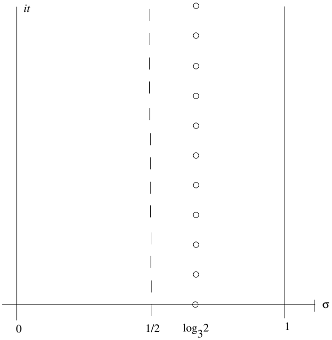
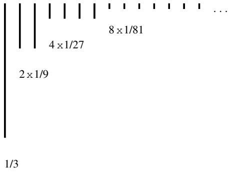
Motivation





Lapidus, van Frankenhuijsen '97, '06, '13

If $\mathcal{L}$ is measurable, then $\mathscr{M} = 2^{1-D_{\mathcal{L}}} \dfrac{\mathrm{res}(\zeta_{\mathcal{L}}(s); D_{\mathcal{L}})}{D_{\mathcal{L}}(1-D_{\mathcal{L}})}$.

Counting boxes

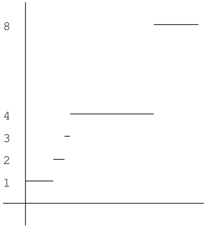


Figure: An example of $N_B(A, x)$ where $x = \varepsilon^{-1}$.

Let $A \subseteq \mathbb{R}^m$ and $\varepsilon > 0$. The *box-counting function* of A, $N_B(A, \varepsilon^{-1})$, is the maximum number of disjoint closed balls $B(a, \varepsilon)$ with centers $a \in A$ of radius ε. We consider the range of $N_B(A, \varepsilon^{-1})$ to be a strictly increasing (with $x = \varepsilon^{-1}$) sequence of positive integers denoted $(M_n)_{n=1}^{\infty}$.

A part of the motivation

For simplicity's sake, we assume throughout that the box-counting dimension of A, $D := \dim_B A$, exists.

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$$\frac{N_B(A, x)}{x^D} \approx \frac{\mathrm{vol}^m(A_\varepsilon)}{\varepsilon^{m-D}}. \qquad (1)$$

Box-counting content and measurability

Dettmers, Giza, Knox, Morales, R. '17

The *upper* and *lower box-counting contents* of a bounded set $A \subseteq \mathbb{R}^m$ are defined, respectively, by

$$\mathscr{B}^*(A) := \limsup_{x\to\infty} \frac{N_B(A,x)}{x^D}, \qquad \mathscr{B}_*(A) := \liminf_{x\to\infty} \frac{N_B(A,x)}{x^D}, \qquad (2)$$

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where $D := \dim_B A$.

If $0 < \mathscr{B}^*(A) = \mathscr{B}_*(A) < \infty$, then A is *box-counting measurable* and the *box-counting content* of A is given by

$$\mathscr{B}(A) := \lim_{x\to\infty} \frac{N_B(A,x)}{x^D}. \tag{3}$$

The unit interval $[0, 1]$ is box-counting measurable

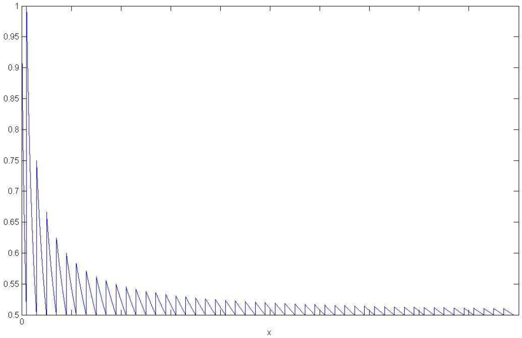


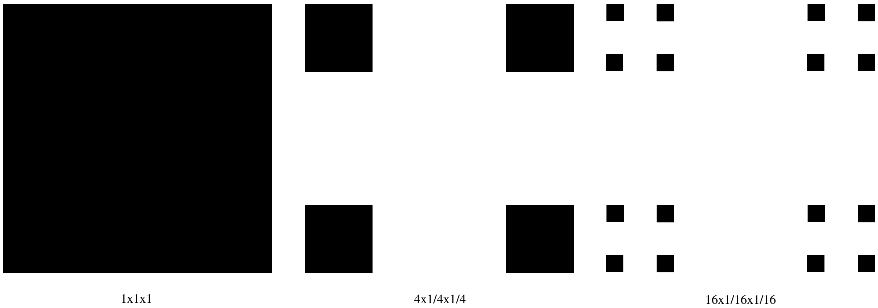
Figure: A plot of $N_B([0,1],x)/x$. Note that $\mathscr{B}([0,1]) = 1/2$.

Hey John...

Show everyone Kalin's movies and ask:

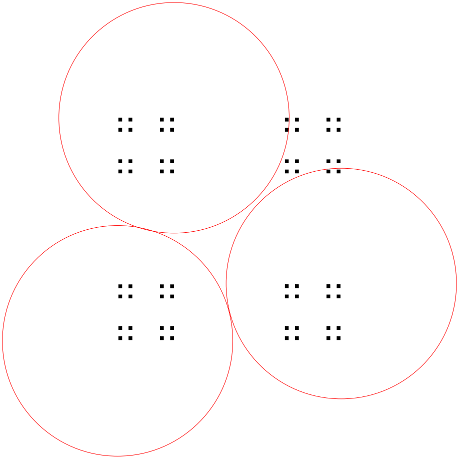
"Which is measurable and which is not?"

The "Quarter Fractal" Q



For the self-similar set Q, nicknamed the "Quarter Fractal", we have $M_1 = 1$ (which is always the case for any bounded set $A \subseteq \mathbb{R}^m$) and $M_2 = 2$.

$M_3 = N_B(Q, (\sqrt{17}/8)^{-1}) = 3$



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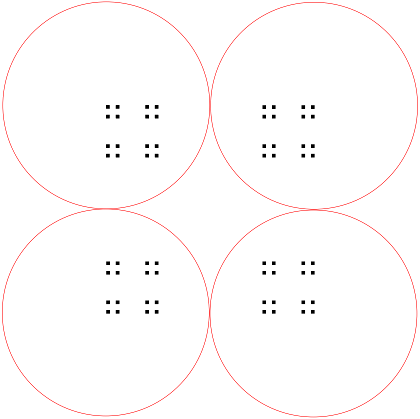
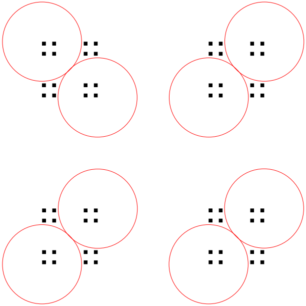


$$M_4 = N_B(Q, (1/2)^{-1}) = 4$$

$M_5 = N_B(Q, (\sqrt{2}/8)^{-1}) = 8 = 4M_2$



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Fractal strings, zeta functions, complex dimensions

Lapidus, van Frankenhuijsen '97, '06, '13

A *fractal string* $\mathcal{L}$ is a sequence of positive real numbers such that

$$\mathcal{L} = (\ell_j)_{j=1}^{\infty} \quad \text{``=''} \quad \{l_n : l_n \text{ distinct with multiplicity } m_n, n \in \mathbb{N}\} \tag{4}$$

where the *lengths* ℓ_j satisfy $0 < \ell_{j+1} \leq \ell_j \; \forall j$ and $\ell_j \to 0$.

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The *dimension* $D_{\mathcal{L}}$, *geometric zeta function* $\zeta_{\mathcal{L}}$, and *complex dimensions* $\mathcal{D}_{\mathcal{L}}$ of $\mathcal{L}$ are respectively given by

$$D_{\mathcal{L}} := \inf\left\{t \in \mathbb{R} : \sum \ell_j^t < \infty\right\}, \tag{5}$$

$$\zeta_{\mathcal{L}}(s) := \sum \ell_j^s = \sum m_n l_n^s, \tag{6}$$

$$\mathcal{D}_{\mathcal{L}}(W) := \{\omega \in W \subseteq \mathbb{C} : \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}, \tag{7}$$

where $\text{Re}(s) > D_{\mathcal{L}}$ and W is a suitable open region. If $W = \mathbb{C}$, we write $\mathcal{D}_{\mathcal{L}}$ for $\mathcal{D}_{\mathcal{L}}(W)$.

Geometric counting functions

Lapidus, van Frankenhuijsen '97, '06, '13

For $x > 0$, the *geometric counting function* of a fractal string $\mathcal{L}$ is given by

$$N_{\mathcal{L}}(x) := \#\{j \in \mathbb{N} : \ell_j^{-1} \leq x\} = \sum_{n \in \mathbb{N},\, l_n^{-1} \leq x} m_n. \quad (8)$$

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Theorem (Lapidus, van Frankenhuijsen '97, '06, '13)

For $\mathrm{Re}(s) > D_{\mathcal{L}}$ *we have*

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} m_n l_n^s = s \int_0^{\infty} N_{\mathcal{L}}(x) x^{-s-1} dx. \tag{9}$$

Fractals strings from counting boxes

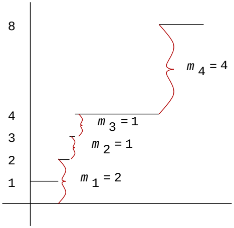

Figure: $N_B(Q, x)$ where $x = \varepsilon^{-1}$.

Lapidus, R., Žubrinić '13

For A and $n \in \mathbb{N}$, let $m_1 := M_2, m_n := M_{n+1} - M_n$ $(n \geq 2)$, and

$$l_n := (\sup\{x \in (0, \infty) : N_B(A, x) = M_n\})^{-1}. \tag{10}$$

Box-counting zeta functions and complex dimensions

Lapidus, R., Žubrinić '13

The *box-counting fractal string* $\mathcal{L}_B$ of A is given by

$$\mathcal{L}_B := \{l_n : l_n \text{ has multiplicity } m_n, n \in \mathbb{N}\}. \tag{11}$$

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The *box-counting zeta function, dimension* and *complex dimensions* of A, denoted ζ_B, D_B, and $\mathcal{D}_B$ are respectively given by

$$\zeta_B := \zeta_{\mathcal{L}_B}, \tag{12}$$

$$D_B := D_{\mathcal{L}_B}, \text{ and} \tag{13}$$

$$\mathcal{D}_B := \mathcal{D}_{\mathcal{L}_B}. \tag{14}$$

Results from box-counting fractal strings

Proposition (Lapidus, R., Žubrinić '13)

Let A *be an infinite subset of* $\mathbb{R}^m$ *with box-counting fractal string* $\mathcal{L}_B$ *and box-counting function* $N_B(A,x)$*. Then for* $x \in (l_1^{-1}, \infty) \setminus (l_n^{-1})_{n\in\mathbb{N}}$*,*

$$N_{\mathcal{L}_B}(x) = N_B(A,x). \qquad (15)$$

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Theorem (Lapidus, R., Žubrinić '13)

Let A be a bounded infinite subset of $\mathbb{R}^m$. Then

$$\overline{\dim}_B A = D_B. \tag{16}$$

Counting functions over complex dimensions

Theorem (Lapidus, van Frankenhuijsen '97, '06, '13)

Let $\mathcal{L}$ *be a fractal string such that* $\mathcal{D}_{\mathcal{L}}(W)$ *consists entirely of simple poles. Then, under certain growth conditions on* $\zeta_{\mathcal{L}}$*, we have*

$$N_{\mathcal{L}}(x) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}(W)} \frac{x^{\omega}}{\omega} \mathrm{res}(\zeta_{\mathcal{L}}(s); \omega) + \{\zeta_{\mathcal{L}}(0)\} + R(x), \qquad (17)$$

where $R(x)$ *is an error term of small order and the term in braces is included only if* $0 \in W \backslash \mathcal{D}_{\mathcal{L}}$.

A criterion for box-counting measurability

Theorem (Giza, Knox, Kurianski, Morales, R. '17)

Let $A \subseteq \mathbb{R}^m$ be a bounded set such that $D = \dim_B A$ exists. Suppose ζ_B satisfies certain growth conditions and has a sufficiently nice meromorphic extension. Then the following are equivalent:

1. *D is the only complex dimension with real part $D = D_B$, and it is simple.*
2. *$N_B(A,x) = \mathscr{B} \cdot x^D + o(x^D)$ as $x \to \infty$ for some positive constant $\mathscr{B}$.*
3. *A is box-counting measurable with box-counting content $\mathscr{B}$.*

If any of the above conditions holds, then

$$\mathscr{B} = \mathscr{B}(A) = \frac{\mathrm{res}(\zeta_B(A,s); D)}{D}. \tag{18}$$

The Riemann zeta function?

Example (Lapidus, R., Žubrinić '13)

For the unit interval $I = [0,1] \times \{0\}$, we have $N_B(I,x) = \lceil x/2 \rceil$ (the ceiling function of $x/2$). Hence, the box-counting zeta function of I is

$$\zeta_B(s) = \frac{1}{2^s} + \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \frac{1}{2^s} + \frac{1}{2^s}\zeta(s), \tag{19}$$

where $\zeta(s)$ is the Riemann zeta function.

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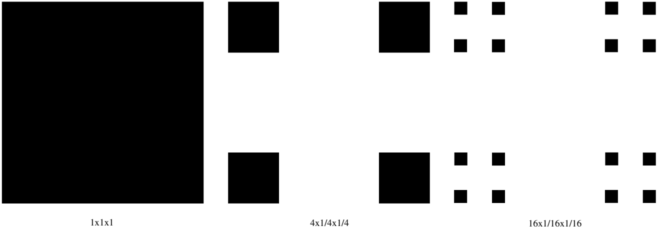
Example (Dettmers, Giza, Knox, Morales, R. '17)

Since $\mathrm{res}(\zeta;1) = 1$ and the meromorphic extension of ζ to $\mathbb{C}$ does not have any other pole with real part 1, we have

$$\mathscr{B}(I) = 1/2 = \mathrm{res}(\zeta_B;1)/1. \tag{20}$$

Strong separation

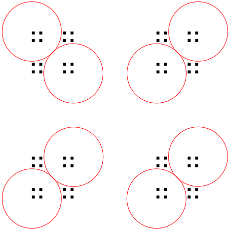


Let $\mathbf{\Phi} = \{\varphi_j\}_{j=1}^N$ be a self-similar system with attractor F and scaling ratios $\mathbf{r} = (r_j)_{j=1}^N$. We say F is *strongly separated* if the $\varphi_j(F)$ are pairwise disjoint and

$$\delta := \sup\{\alpha : d(x,y) > \alpha, x \in \varphi_j(F), y \in \varphi_k(F), j \neq k\} \tag{21}$$

is finite and positive.

Box-counting functions and strong separation



Lemma (Lalley '88)

Let $F \subseteq \mathbb{R}^m$ be a strongly separated self-similar set. Then for any $x > 0$,

$$N_B(F,x) = \sum_{j=1}^{N} N_B(F, r_j x) + L(x) \tag{22}$$

where $L(x)$ is an integer valued step function that vanishes for $x > \delta^{-1}$.

Box-counting zeta functions and strong separation

Theorem (Sargent '14)

Suppose Φ *is a strongly separated self-similar system with attractor* $F \subseteq \mathbb{R}^m$ *and scaling ratios* $r_j, n = 1, \ldots, N$. *Let* $\mathcal{L}_B$ *be the box-counting fractal string of* F *with first length* ℓ_1. *Then*

$$\zeta_B(s) = \frac{h(s)}{1 - \sum_{j=1}^{N} r_j^s} \tag{23}$$

where

$$h(s) := \ell_1^s \left(\sum_{j=1}^{N} (1 - r_j^s) \right) + s \int_{\ell_1^{-1}}^{\infty} L(x) x^{-s-1} dx \tag{24}$$

is an entire function.

ζ_B and $\mathcal{D}_B$ of the Quarter Fractal Q

Example (Lapidus, R., Žubrinić '13; Sargent '14)

For Q as above we have

$$\zeta_B(s) = \left(\frac{\sqrt{2}}{2}\right)^s + \frac{\left(\sqrt{2}/2\right)^s + \left(\sqrt{17}/8\right)^s + (1/2)^s}{1 - 4 \cdot 4^{-s}}. \quad (25)$$

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$$\mathcal{D}_B = \left\{1 + \frac{2\pi i z}{\log 4} : z \in \mathbb{Z}\right\}. \tag{26}$$

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Q is *not* box-counting measurable since it has nonreal box-counting complex dimensions ω with real part Re$\omega = 1$.

Lattice approximation of nonlattice self-similar sets

Theorem (Dettmers, Giza, Knox, Morales, R. '17)

Suppose $\mathbf{\Phi}$ is a strongly separated nonlattice self-similar system on $\mathbb{R}^m$ with attractor F, scaling ratios $\mathbf{r} = (r_j)_{j=1}^N$, and box-counting complex dimensions $\mathcal{D}$. Additionally, assume $h(s) = 0$ if and only if $\sum_{j=1}^N r_j^s = 1$. Then there is a sequence of lattice self-similar systems $(\mathbf{\Phi}_M)_{M=1}^\infty$ with, respectively, attractor F_M, scaling ratios $\mathbf{r}_M = (r_{M,j})_{j=1}^N$, and box-counting complex dimensions $\mathcal{D}_M$ such that the following hold as $M \to \infty$:

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Under additional hypotheses,

- $\mathcal{D}_M \to \mathcal{D}$ *in the sense described in Chapter 3 of Lapidus, van Frankenhuijsen '97, '06, '13.*

A (generic) nonlattice self-similar set

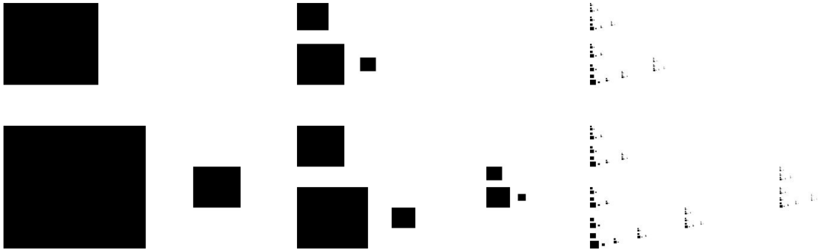

Figure: Constructing a strongly separated (generic) nonlattice self-similar set S with scaling ratios $1/2, 1/3$, and $1/6$.

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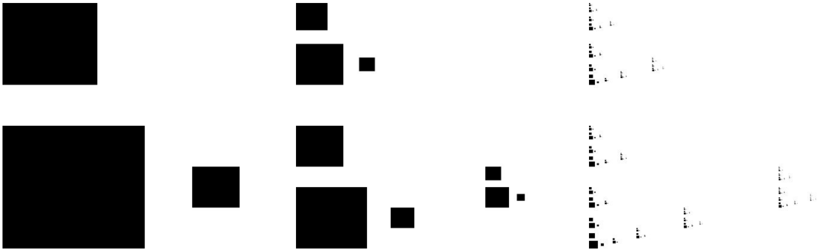
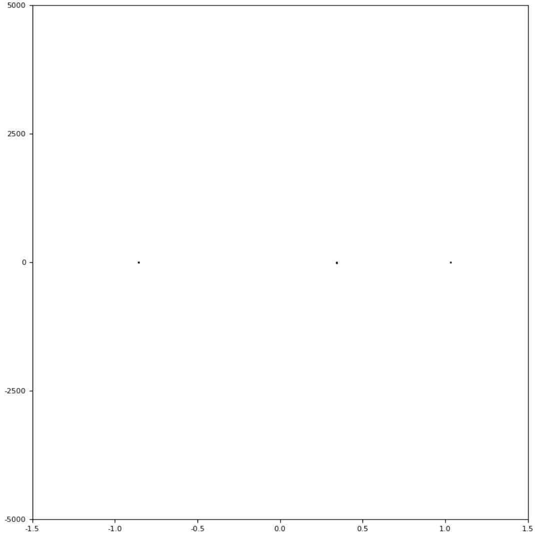


Figure: Constructing a strongly separated (generic) nonlattice self-similar set S with scaling ratios $1/2, 1/3$, and $1/6$.

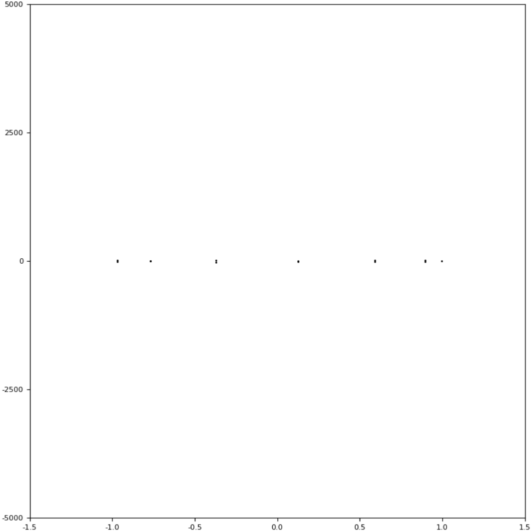
Let $\mathcal{D}$ denote the set of roots of the corresponding Moran equation

$$\sum_{j=1}^{3} r_j^s = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{6^s} = 1. \qquad (27)$$

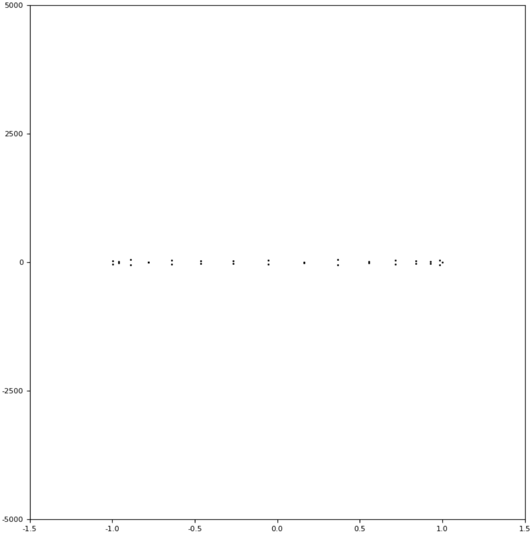
A first lattice approximation of $\mathcal{D}$



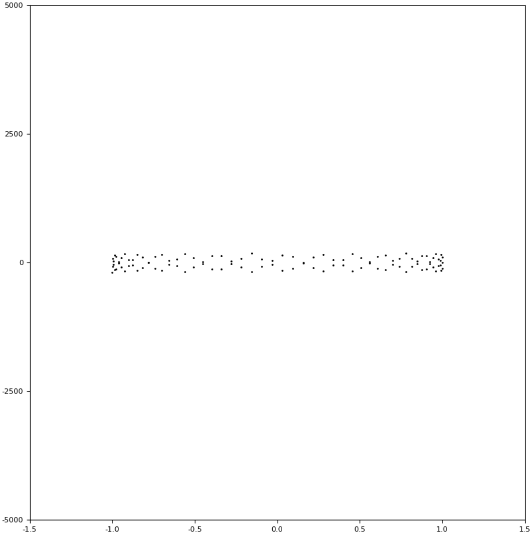
A second lattice approximation of $\mathcal{D}$



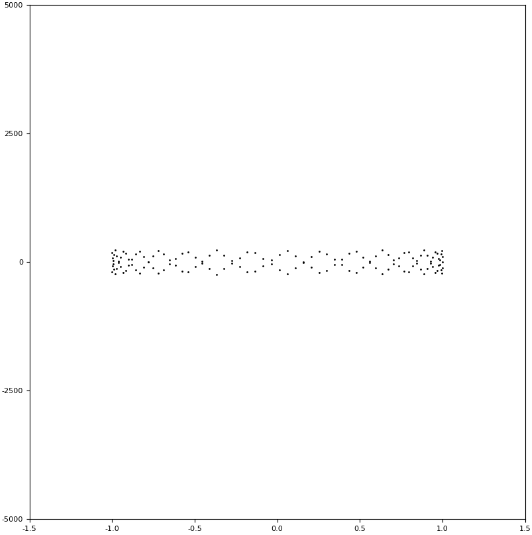
A third lattice approximation of $\mathcal{D}$



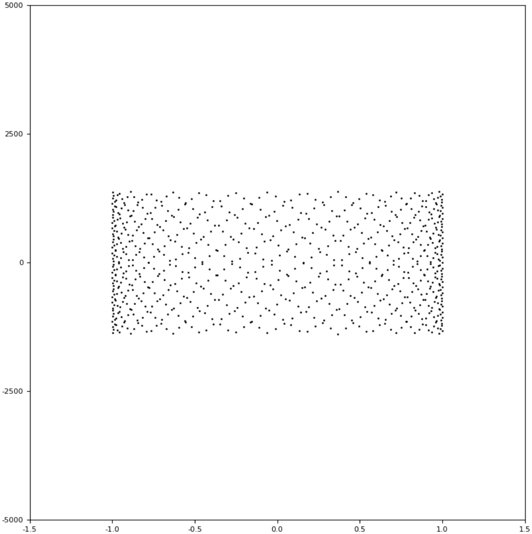
A fourth lattice approximation of $\mathcal{D}$



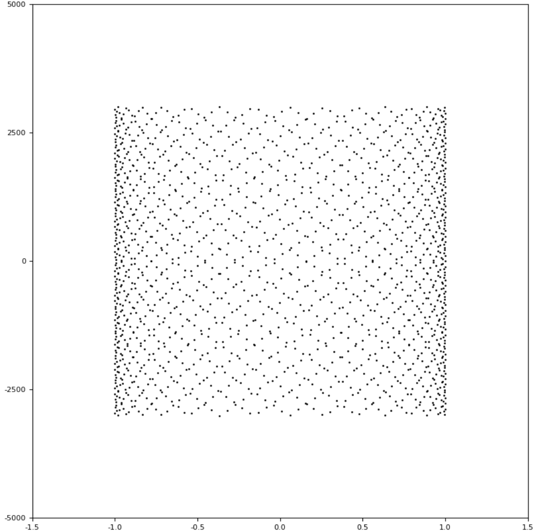
A fifth lattice approximation of $\mathcal{D}$



A sixth lattice approximation of $\mathcal{D}$



A final lattice approximation of $\mathcal{D}$



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