

Degenerate harmonic structures on fractal graphs.

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Analysis on fractals via analysis on graphs.

The motivation comes from analysis on fractals. The pointwise formula connects the Laplace operator on the fractal with the discrete graph Laplacians.

$$\Delta u(x) = \lim_{m \rightarrow \infty} \lambda^m \Delta_m u(x)$$

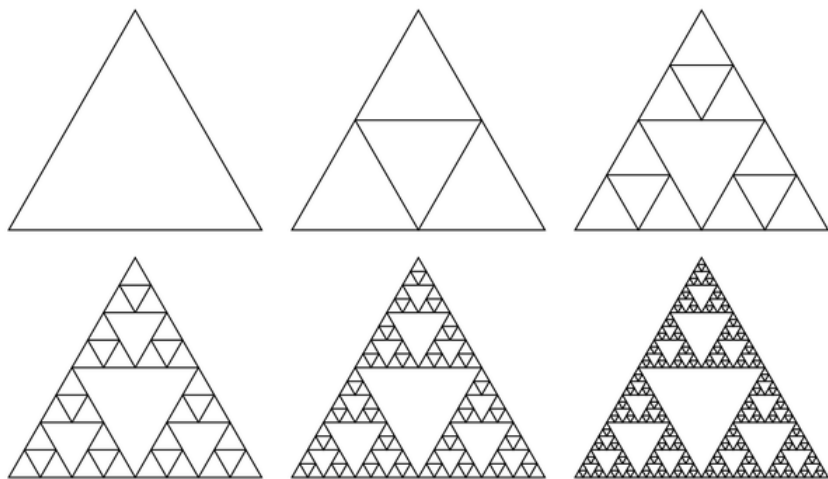
Harmonic functions satisfy

$$\Delta u = 0.$$

Question

Can we get any information on harmonic functions by studying them at the graph level?

Pre-fractal graphs.



A couple definitions.

We can define the energy of a function on these graphs as

$$\mathcal{E}_m(u) = \left(\frac{1}{r}\right)^m \sum_{y \sim_m x} (u(y) - u(x))^2$$

and the combinatorial graph Laplacian as

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(x) - u(y))$$

or equivalently as

$$\Delta_m = D_m - A_m$$

where D_m and A_m are the degree and adjacency matrices of the appropriate graph level approximations.

Harmonic functions on pre-fractal graphs.

Equivalent criteria for the Dirichlet problem on the graphs:

- Minimize energy.
- $\Delta_m h(x) = 0$ for $x \notin V_0$.
- Satisfy mean-value property for $x \notin V_0$, i.e

$$h(x) = \frac{1}{\deg(x)} \sum_{y \sim_m x} h(y)$$

- They have a probabilistic interpretation.

Question

Can we utilize the self-similarity of K to solve the Dirichlet problem algorithmically?

Harmonic extension matrices.

If $|V_0| = k$, define A_i to be the harmonic extension matrices such that

$$\begin{pmatrix} h \circ F_i(q_1) \\ \cdot \\ \cdot \\ \cdot \\ h \circ F_i(q_k) \end{pmatrix} = A_i \begin{pmatrix} h(q_1) \\ \cdot \\ \cdot \\ \cdot \\ h(q_k) \end{pmatrix}$$

- For the cell $F_w K$ we can use $A_w = A_{w_m} \cdots A_{w_2} A_{w_1}$.
- Since constants remain constants, they are stochastic matrices.
- The second eigenvalue of the matrices of boundary points is the renormalization constant.

Example on SG_2 .

$$\begin{pmatrix} h \circ F_i(q_1) \\ h \circ F_i(q_2) \\ h \circ F_i(q_3) \end{pmatrix} = A_i \begin{pmatrix} h(q_1) \\ h(q_2) \\ h(q_3) \end{pmatrix}$$

with

$$A_1 = \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \quad A_2 = \frac{1}{5} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix} \quad A_3 = \frac{1}{5} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

Question

Are these A_i invertible matrices?

Let h a non-constant harmonic function. Define the energy measure of h to be

$$\nu_h(F_w K) = r^{-|w|} \mathcal{E}(h \circ F_w)$$

where r is the renormalization constant. If they are not invertible then non-constant harmonic functions can be locally constant on some cell. If such an h exists, then ν_h gives to a cell zero measure.

- Energy measures lack self-similarity but have other good properties (Kusuoka measure/Laplacian).
- In \mathbb{R}^d , non-constant harmonic functions cannot be locally constant.

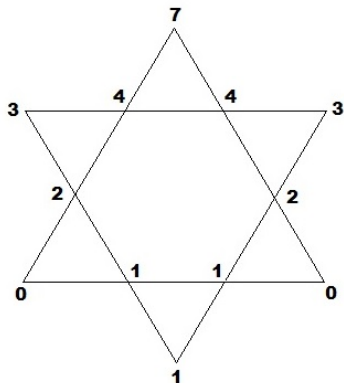
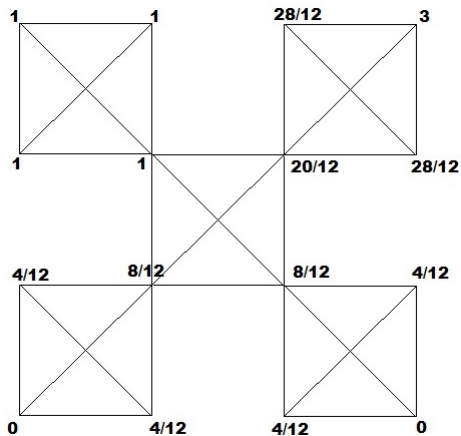
Motivation.

A lot of results in the field require the non-degeneracy condition.

- Absolute continuity of different energy measures.
- Radial distribution of coefficients related to Radon-Nikodym averages.
- Random matrices and derivatives of p.c.f fractals.

Answer

Usually not. Such a harmonic structure is called degenerate.



Question

Can we give a criterion for when a finitely ramified self-similar set has a degenerate harmonic structure?

Question

Can we find self-similar sets that have a non-degenerate harmonic-structure?

- These matrices are created only on the first graph approximation, so it suffices to study them only on these graphs.

Answer

“Bad” graph connectivity properties give us degeneracies.

Definition

A graph is k -vertex connected if it cannot become disconnected by removing at most $k - 1$ vertices. Equivalently, for every pair of vertices there are at least k vertex independent paths connecting them. Its vertex connectivity is the largest k such that it is k -vertex connected.

Let K be a finitely ramified self-similar set and \tilde{G}_1 be the modified G_1 graph by adding extra edges connecting all boundary points.

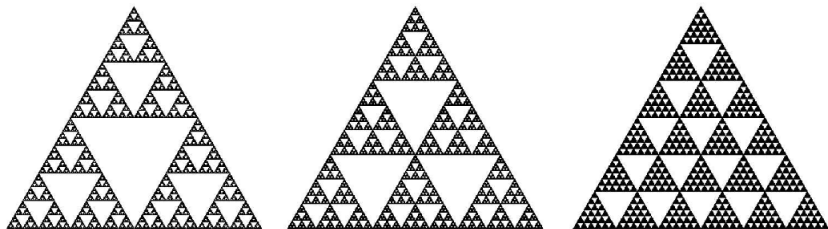
Proposition

If \tilde{G}_1 has vertex connectivity less than $|V_0|$, then the self-similar set K has a degenerate harmonic structure.

Proof.

If we have connectivity $k < |V_0|$ then we can remove vertices v_1, \dots, v_k making the graph have at least two connected components. A cell must necessarily be included in $C \cup \{v_1, \dots, v_k\}$ where C is a connected component not containing any boundary points. Then since $\dim \mathbb{R}^{|V_0| \times 1} > \dim \mathbb{R}^{k \times 1}$ we can create a non-constant harmonic function which is constant on $C \cup \{v_1, \dots, v_k\}$ and thus on some cell.

Sierpiński gaskets of level k .



- SG_k has $\frac{k(k+1)}{2}$ harmonic extension matrices.
- Can be generalized to higher dimensions.

First graph approximations.

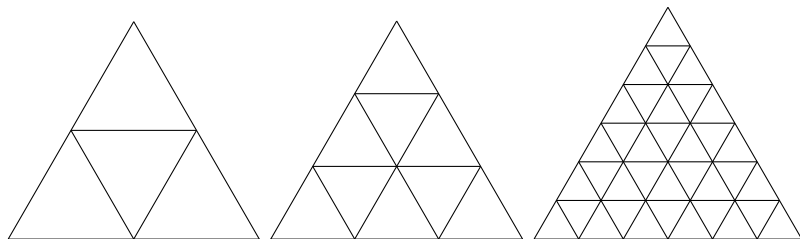


Figure: The G_1 graph of SG_2 , SG_3 and SG_6 .

A few cases.

- For SG_2 , $A_1 = \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ with eigenvalues $\frac{1}{5}, \frac{3}{5}, 1$
- For SG_3 ,

$$A_1 = \frac{1}{15} \begin{pmatrix} 15 & 0 & 0 \\ 8 & 4 & 3 \\ 8 & 3 & 4 \end{pmatrix} \quad A_4 = \frac{1}{15} \begin{pmatrix} 8 & 4 & 3 \\ 4 & 8 & 3 \\ 5 & 5 & 5 \end{pmatrix}$$

with eigenvalues $\lambda = \frac{1}{15}, \frac{7}{15}, 1$ and $\lambda = \frac{2}{15}, \frac{4}{15}, 1$

- They are all invertible for all SG_k with $k \leq 50$ (numerical calculations, Hino 2009).

Hino's conjecture/results.

Conjecture

For all $k \geq 2$, the harmonic extension matrices of SG_k are invertible.

Theorem

For every non-constant harmonic function h the energy measure ν_h is minimally energy dominant. In particular for every two non-constant harmonic functions h_1, h_2 the energy measures ν_{h_1}, ν_{h_2} are mutually absolutely continuous.

Geometric representation of graphs.

- A graph embedding in a surface is a representation of the graph to the surface associating vertices to points and edges to simple arcs so that the arcs do not include any other vertices or intersect with each other except at their common end points.
- A graph is planar if it can be embedded in the plane.
- Kuratowski's theorem states that a graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

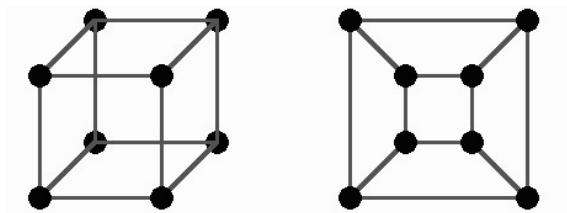
Question

How can we draw a planar graph?

- The answer to this question provides the machinery to answer the conjecture.

Rubber band representation.

Let G a connected planar graph and take S to be the vertices bounding a face of it. Fix the vertices of S in \mathbb{R}^2 . Think of the edges as ideal rubber bands satisfying Hooke's Law and let the other vertices settle in equilibrium. This is the rubber band representation of G in \mathbb{R}^2 with S nailed/fixed.



How to draw a graph.

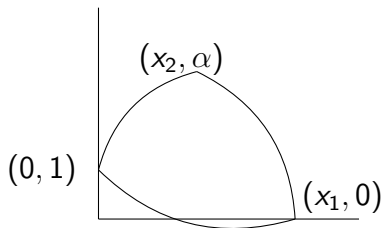
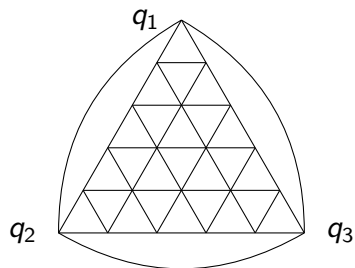
- Equilibrium has minimum energy. All free vertices are at the barycenter of its neighbors.
- Each coordinate function is harmonic on $V \setminus S$.

Theorem (Tutte 1963)

If G is a simple 3-connected planar graph, then its rubber band representation with respect to any of its faces is an embedding of G in the plane.

Proof of non-degeneracy for SG_k .

- It is obviously planar.
- Extra boundary edges do not perturb Laplace's equation and make the graph at least 3-connected.



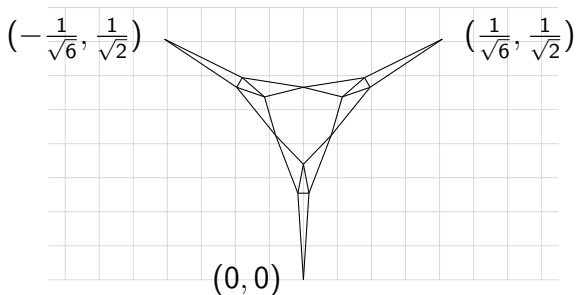
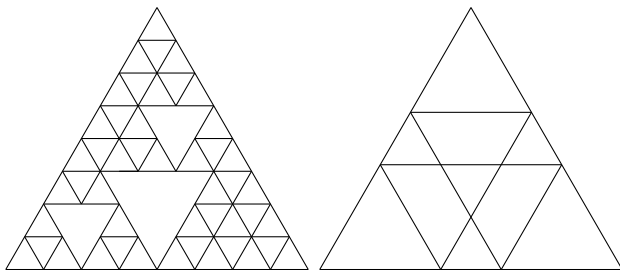


Figure: A barycentric embedding of the second graph approximation of SG_2 .

- Kusuoka measure becomes now renormalized energy of the graph representation.

Variations.



What about higher dimensions?

- Hino's conjecture is also stated for SG_k^d for all $d \geq 2$.

Question

What conditions along with $|V_0|$ or higher vertex connectivity are also sufficient?

- A similar approach works only for weighted edges.

Thank you for your attention!