Ollivier Ricci curvature for general graph Laplacians

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joint work with Florentin Münch (University of Postdam)

Introduction

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Idea: Extend the notion of Ricci curvature introduced by Ollivier and modified by Lin-Liu-Yau to the case of general (possibly unbounded) graph Laplacians. In this setting new phenomena, which do not appear in the case of bounded operators, such as stochastic completeness can be explored.

Outline

- Framework (weighted graphs, Laplacians, curvature)
- Semigroup characterization
- Criteria for stochastic completeness
- Criteria for finiteness

Setting - weighted graphs

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If $w(x, y) > 0$ we say that x and y are *connected* by a weighted edge or are *neighbors* and write $x \sim y$ in this case. We call $G = (V, w, m)$ a *weighted graph*.

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We let

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y \in V} w(x,y)$$

denote the *weighted degree* of a vertex x.

Connectedness, metric and Laplacian

We assume that w is *connected* in the sense that for any two vertices x and y there exists a sequence $(x_k)_{k=0}^n$ with $x_0 = x$, $x_n = y$ and $x_k \sim x_{k+1}$ for $k = 0, 1, \ldots, n-1$. Such a sequence is called a *path* connecting x and y.

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We let $C(V) = \{f : V \to \mathbb{R}\}$ and let $\Delta : C(V) \to C(V)$ be given by

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in V} w(x, y)(f(y) - f(x)).$$

Δ is called the *Laplacian* associated to the weighted graph.

Examples: (Normalized) Graph Laplacian

We give two examples based on standard edge weights and two commonly appearing measures.

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Let $w(x, y) \in \{0, 1\}$.

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$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)).$$

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1. If $m = 1$, then Δ is called the *graph Laplacian* given by
$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)).$$
2. If $m(x) = d_x := \sum_{y \in X} w(x, y) = |\{y \mid y \sim x\}|$, then Δ is called the *normalized Laplacian* given by
$$\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} (f(y) - f(x)).$$

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By a direct calculation, one gets that

$$m_x^\varepsilon(y) = \begin{cases} 1 - \varepsilon \mathrm{Deg}(x) & : y = x \\ \varepsilon \frac{w(x,y)}{m(x)} & : \text{otherwise} \end{cases}$$

where $\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y \in V} w(x,y)$. In particular, if ε is small enough, then m_x^ε is a probability measure.

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where $\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y \in V} w(x,y)$. In particular, if ε is small enough, then m_x^ε is a probability measure.
For probability measures, the transportation distance can be defined as follows:

$$W(m_x^\varepsilon, m_y^\varepsilon) = \sup_{f \in Lip(1)} \sum_{z \in V} f(z)(m_x^\varepsilon(z) - m_y^\varepsilon(z))\}$$

where $Lip(1) = \{f \in C(V) \mid |f(u) - f(v)| \leq d(u,v)$ are the functions with Lipshitz constant 1.

Measure, transportation distance, curvature – continued

A direct calculation then gives that

$$W(m_x^\varepsilon, m_y^\varepsilon) = \sup_{f \in Lip(1)} (f(x) + \varepsilon \Delta f(x) - (f(y) + \varepsilon \Delta f(y)))$$
$$= \sup_{f \in Lip(1)} \nabla_{xy}(f + \varepsilon \Delta f) d(x,y)$$

where $\nabla_{xy} f := (f(x) - f(y))/d(x,y)$.

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where $\nabla_{xy} f := (f(x) - f(y))/d(x,y)$.
For $x \neq y$, let

$$\kappa_\varepsilon = 1 - \frac{W(m_x^\varepsilon, m_y^\varepsilon)}{d(x,y)}$$

and define

$$\kappa(x,y) := \lim_{\varepsilon \to 0^+} \frac{\kappa_\varepsilon}{\varepsilon}$$

as the *Ricci curvature* at vertices x and y.

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Ollivier (2009) introduces this idea for Markov chains on metric spaces and looks at it for the special case of unweighted graphs for $\varepsilon = 0$ and $\varepsilon = 1/2$. Lin-Liu-Yau (2011) introduce this formula for the normalized graph Laplacian. In this case, $\text{Deg} = 1$.

Semigroup characterization

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$$\begin{cases} \Delta u(x,t) &= \partial_t u(x,t) \quad x \in V,\ t \geq 0 \\ u(x,0) &= f(x) \quad\quad\quad\ x \in V. \end{cases}$$

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We say that a graph is *Feller* if P_t maps functions vanishing at infinity to functions vanishing at infinity.
We let $\|\nabla f\|_\infty := \sup_{x \neq y} |\nabla_{xy} f|$ denote the Lipshitz constant of a function. We write $Ric(G) \geq K$ if $\kappa(x,y) \geq K$ for all $x, y \in V$.
With these notations, we can state our first result as a characterization of lower curvature bounds for graphs which satisfy the Feller property.

Semigroup characterization – continued

Theorem

Let G be a weighted graph which satisfies the Feller property. The following statements are equivalent:

(i) $Ric(G) \geq K$.

(ii) *For all bounded functions f and all $t > 0$*

$$\|\nabla P_t f\|_\infty \leq e^{-Kt} \|\nabla f\|_\infty.$$

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This theorem gives an analogue to a result for Riemannian manifolds by Renesse and Sturm (2004). They do not assume the Feller property; however, in the manifold case a lower Ricci curvature bound immediately implies the Feller property which is not true for graphs.

Calculating the Ricci curvature

Our next result is a limit-free formula for the curvature which makes the curvature easy to calculate in some cases.

Theorem

Let $x \neq y$, then

$$\kappa(x,y) = \inf_{\substack{f \in Lip(1) \\ \nabla_{yx} f = 1}} \nabla_{xy} \Delta f.$$

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Idea of proof: To show that $\frac{1}{\varepsilon}\kappa_\varepsilon(x,y) \leq \inf_{\substack{f \in Lip(1) \\ \nabla_{yx} f = 1}} \nabla_{xy}\Delta f$ is easy.

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Example

Line graphs Let $V = \mathbb{N}_0$ with $w(m,n) = 0$ if $|m-n| \neq 1$. If $f(n) = n$ and $r < R$, then

$$\kappa(r,R) = \nabla_{rR}\Delta f = \frac{\Delta f(r) - \Delta f(R)}{R - r}.$$

Laplace comparison

We now fix a reference vertex x_0 and let S_r denote the sphere of radius r about x_0. We let

$$\kappa(r) = \min_{y \in S_r} \max_{\substack{x \in S_{r-1} \\ x \sim y}} \kappa(x, y)$$

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Our main technical tool for the results below is the following Laplace comparison theorem.

Theorem

If $f(x) = d(x, x_0)$, then

$$\Delta f(x) \leq \mathrm{Deg}(x_0) - \sum_{r=1}^{f(x)} \kappa(r).$$

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Idea of proof: by induction on R and using the formula for computing the curvature above.
Note: The inequality above is sharp on line graphs.

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Idea of proof: Use the Khas'minskii criterion for stochastic completeness along with the Laplace comparison above.
Note: the result is sharp as there exist stochastically incomplete line graphs with $\kappa(r) \geq -(\log r)^{1+\varepsilon}$ for any $\varepsilon > 0$.

Finiteness

We can also give an improved diameter bound which then implies finiteness of the graph if the weighted degree is bounded.

Theorem

If there exists an R *with,*

$$\sum_{r=1}^{R} \kappa(r) > \mathrm{Deg}(x_0) + \max_{x \in S_R} \mathrm{Deg}(x),$$

then $diam(G) < 2R$ *where* $diam(G) = \sup_{x,y \in V} d(x,y)$.

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Idea of proof: use the Laplace comparison and argue by contradiction.

Finiteness – continued

As an immediate corollary, we get the following statement.

Corollary

Suppose that $\sup_{x \in V} \mathrm{Deg}(x) < \infty$ *and* $\sum_r \kappa(r) = \infty$, *then the graph is finite.*

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Suppose that $\sup_{x \in V} \mathrm{Deg}(x) < \infty$ *and* $\sum_r \kappa(r) = \infty$, *then the graph is finite.*

Note: the result on the diameter is sharp. Furthermore, there exist infinite graphs with uniformly positive curvature. However, for such graph the weighted degree is unbounded.

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Thanks

Thank you for your attention!