A local time scaling exponent for compact metric spaces

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Local Hausdorff Dimension

- Let (X, d) a metric space.
- If $A \subset B \subset X$ then $\dim_H(A) \leq \dim_H(B)$[1].
- So define the local Hausdorff dimension α by...

Definition

$\alpha(x) := \lim_{r \to 0^+} \dim_H(B_r(x)) = \inf_{r>0} \dim_H(B_r(x)).$

[1] $\dim_H$ means the Hausdorff dimension

Carathéodory construction of a metric outer measure

- For $\delta > 0$ and $A \subset X$ let $\mathcal{C}_\delta(A)$ be the collection of all covers of A by an at most countable number of subsets of X of diameter at most δ.
- Then let $\mu^*(A) = \sup_{\delta>0} \inf\{\sum_{U\in\mathcal{U}} \mathrm{diam}(U)^{\dim_H(U)} \mid \mathcal{U} \in \mathcal{C}_\delta(A)\}$.
- If $\mathcal{M}^* := \{A \subset X \mid \forall E \subset X\ [\ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)\ ]\ \}$, then $\mathcal{M}^*$ is a σ-algebra containing the Borel measurable sets and $\mu := \mu^*|_{\mathcal{M}^*}$ is a complete measure.
- We call μ the local Hausdorff measure of variable dimension $\alpha(\cdot)$.

Variable Ahlfors Regularity

By $f \approx g$ we mean there exists a constant $C > 0$ (independent of the arguments of f and g) such that $\frac{1}{C} f \leq g \leq Cg$.

Definition

If $Q : X \to (0, \infty)$, a borel measure ν is called variable Ahlfors $Q(\cdot)-$regular if

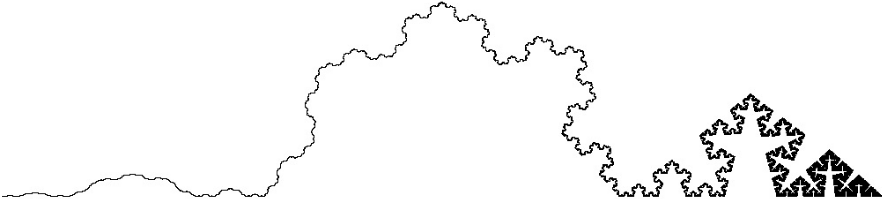
$$\nu(B_r(x)) \approx r^{Q(x)}.$$

Theorem

[a] Let X be a compact metric space. Then if ν is an Ahlfors Q-regular Borel measure then $Q = \alpha$ and $\nu \approx \mu$.

[a] J. Dever, *Local Hausdorff Measure*, ArXiv e-prints (2016).

A Koch curve of continuously varying local dimension.[2]



- Constructed by recursively varying the angles of the generator.
- Any constant dimensional Hausdorff measure gives measure 0 or ∞. But the μ measure is positive and finite.

[2] Laurent Nottale *Fractal space time and microphysics: towards a theory of scale relativity*, Petteri Harjulehto, Peter Hästö, and Visa Latvala, *Sobolev embeddings in metric measure spaces with variable dimension*, Mathematische Zeitschrift **254** (2006), no. 3, 591–609.[8]

Time scaling exponent: discrete approach

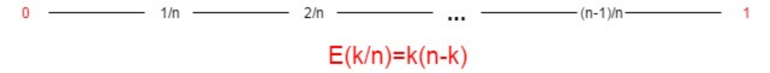
From now on: X connected, compact, Ahlfors $\alpha(\cdot)-$regular.

- An $\epsilon-$net on X is a set $N \subset X$ such that $\cup_{x \in N} B_\epsilon(x) = X$ and if $x, y \in X$ with $x \neq y$ then $d(x, y) \geq \epsilon$.
- We way define an induced graph G_N with vertex set N in many ways. Some choices for edge relations are as follows:
  - Given $c > 0$ set $x \sim y$ if $B_{c\epsilon}(x) \cap B_{c\epsilon}(y) \neq \emptyset$.
  - For $x \in N$ let $T(x) = \{z \in X \mid d(x, z) \leq d(y, z) \text{ for all } y \in N\}$ be the Voronoi tile of x. Set $x \sim y$ it $T(x) \cap T(y) \neq \emptyset$.
- For $x \in N$ let $\deg(x) = \#\{y \in N \mid y \sim x\}$. Let D_N be the *degree matrix* and A_N the *adjacency matrix* of G_N ...i.e. $D_{N_{x,y}} = \deg(x)\delta_{x,y}$ and $A_{N_{x,y}} = 1$ if $x \sim y$ and 0 otherwise.
- Define a random walk $(X_k)_{k=1}^{\infty}$ by the transition matrix $P_N = {D_N}^{-1} A_N$, $P_{N_{x,y}} = p_N(x, y) = \frac{1}{\deg(x)} A_{N_{x,y}}$.[2]

Time scaling exponent: discrete approach

- Let $B = B_R(x_0)$ be a ball. Let $\tau_{B,N} = \inf\{k \mid X_k \notin B\}$.
- Let $E_{B,N}(x) = \mathbb{E}^x \tau_{B,N}$. Then we have the exit time equation $E_{B,N}(x) = 1 + \sum_{y \sim x} p_N(x,y) E_{B,N}(y)$ for $x \in B$ and $E(x) = 0$ for $x \notin B$, i.e. $(I - P_{B,N})E_B = 1$ for $x \in B$, $E(x) = 0$ for $x \notin B$.

Example



Time scaling exponent: discrete approach

Set $E^+_{B,N} = \max_{x \in N} E_{B,N}(x)$. Then let

- $\omega_\beta(B) = \inf_{\delta>0} \sup\{E^+_{B,N}\epsilon^\beta \mid N \text{ an } \epsilon-\text{net}^3 \text{ with } \epsilon < \delta\}$

Note if $\beta' < \beta$ then $E^+_{N,B}\epsilon^\beta \leq \delta^{\beta-\beta'} E^+_{N,B}\epsilon^{\beta'}$ It follows that if $\omega_\beta(B) > 0$ then $\omega_{\beta'}(B) = \infty$ and if $\omega_{\beta'}(B) < \infty$ then $\omega_\beta(B) = 0$. Hence we may define

- $\beta(B) := \sup\{\beta \geq 0 \mid \omega_\beta(B) = \infty\} = \inf\{\beta \geq 0 \mid \omega_\beta(B) = 0\}$.

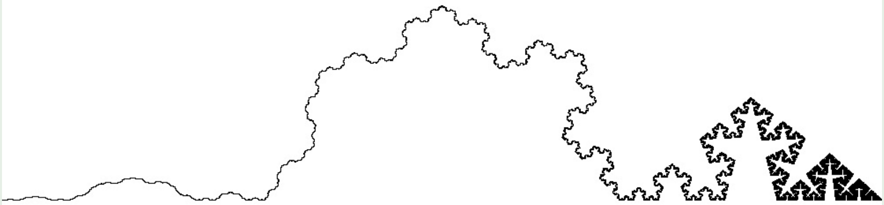
Then if $B' \subset B$, $\beta(B') \leq \beta(B)$.
We define...

Definition

$\beta(x) := \inf_{R>0} \beta(B_R(x)) = \lim_{R \to 0^+} \beta(B_R(x))$.

[3] Another (weaker) possibility is to take $\limsup$ only along a chosen sequence of nested $\epsilon -$ *nets* or dyadic cube approximations with the length scale going to 0.

Example



Here $\beta = 2\alpha$. Hence β can vary continuously as well!

Example

On $\mathbb{R}^n$ with standard euclidean lattice approximations, with the (weaker) definition of β we have $\beta(x) = 2$.

Example

On the Sierpinski gasket the exit time from a ball of graph distance n is of order 5^n. Since length scale is $\frac{1}{2^n}$, (with weaker def.) $\beta(x) = \frac{\log(5)}{\log(2)}$.

Continuous space approach

Since we have an Ahlfors $\alpha(\cdot)-$regular measure μ we may define a random walk at stage r by the transition kernel

$$p_r(x,y) = \frac{1}{\mu(B_r(x))}\chi_{B_r(x)}(y), \quad x,y \in X.$$

Define P_r on $L^2(\mu)$ by $P_r f(x) = \int p_r(x,y)d\mu(y)$. This defines a (continuous space, discrete time) random walk $(Y_k)_{k=0}^{\infty}$.
If $B = B_R(x_0)$ is a ball let $\tau_B = \inf\{k \mid Y_k \notin B\}$.
Let $E_{r,B}(x) = \mathbb{E}^x \tau_{B,r}(x)$.
Again we have the exit time equation
$E_{r,B}(x) = 1 + \int_X p_r(x,y)E_{r,B}(y)d\mu(y)$ for $x \in B$ and $E_{r,B}(x) = 0$ for $x \notin B$. So $(I - P_r)E_{r,B} = 1$ on B and $E_{r,B}(x) = 0$ for $x \notin B$.

Time scale exponent: continuous space approach

Let $E^+_{r,B} = \sup_{x \in B} E_{r,B}$.
Then let $\omega_\beta(B) = \limsup_{r \to 0^+} E^+_{r,B} r^\beta$.
As before there is a critical exponent
$\beta(B) = \sup\{\beta > 0 \mid \omega_\beta(B) = \infty\} = \inf\{\beta > 0 \mid \omega_\beta(B) = 0\}$.

Definition

$\beta(x) := \inf_{R>0} \beta(B_R(x)) = \lim_{R \to 0^+} \beta(B_R(x))$.

Example

Let $B = B_R(0)$ a ball about the origin in $\mathbb{R}^n$ with euclidean norm. Let r small enough so that $B_r(x) \subset B$. Then we have

$$E_r(x) = \left(\frac{n+2}{n}\right) \frac{R^2 - |x|^2}{r^2}.$$

Hence $\beta(x) = 2$ on $\mathbb{R}^n$.

Continuous time re-normalization

Let $\tau_r(x) = r^{\beta(x)}$ for $x \in X$. Let $\Omega = (\mathbb{R}^+ \times X)^{\mathbb{Z}_+}$. For $x = x_0, r > 0$ define a measure on cylinder sets by

- $\mathbb{P}_r^{x_0}(\{\omega \in (\mathbb{R}^+ \times X)^{\mathbb{Z}_+} \mid \omega(k) \in I_k \times A_k \text{ for } k = 1, ..., n\}) = \left(\prod_{k=1}^n \int_{I_k} \int_{A_k} \frac{e^{-t_k/\tau_r(x_{k-1})}}{\tau_r(x_{k-1})} p_r(x_{k-1}, x_k)\right) dt_n d\mu(x_n) ... dt_1 d\mu(x_1)$.[4]

This is Kolmogorov consistent. By the Kolmogorov Extension Theorem we may extend to a prob. measure $\mathbb{P}_r^x$ on Ω with product σ-algebra. For basepoint $x = x_0$ set $\omega(0) = (0, x_0)$. Then for $t \geq 0, \omega \in \Omega$, let

- $\hat{t}(\omega) = \inf\{k \mid \sum_{j=0}^k \omega(k)_1 \geq t\}$.

Then define $(X_t^{(r)})_{t \geq 0}$ by

- $X_t^{(r)}(\omega) = \omega(\hat{t}(\omega))_2$.

[4] A measure with local exponential waiting times in the case of a graph was considered by Bellissard in [3]. A formula for the generator was also given there.

Generator

Then $(X_t^{(r)})_{t\geq 0}$ has generator $\mathcal{L}_r$ on $L^2(\mu)$ defined by

- $\frac{d}{dt}\Big|_{t=0} \mathbb{E}^x f(X_t) = \mathcal{L}_r f(x) := \frac{1}{r^{\beta(x)}\mu(B_r(x))} \int_{B_r(x)} (f(x) - f(y))d\mu(y)$. [3]
- There is an equilibrium probability measure ν_r with density $d\nu_r(x)/d\mu(x) := \frac{r^{\beta(x)}\mu(B_r(x))}{Z_r}$, where Z_r is the normalization factor $Z_r = \int_X r^{\beta(z)}\mu(B_r(z))d\mu(z)$.
- Then $\mathcal{L}_r$ is self-adjoint on $L^2(\nu_r)$.
- Let $\tau_{B,r} := \inf\{t \mid X_t^{(r)} \notin B\}$.
- For $x \in X$ let $\phi_{r,B}(x) := \mathbb{E}^x \tau_{B,r}$. Let $\phi_{r,B}^+ := \sup_{y \in B} \phi_{r,B}(y)$.

Variable Time Regularity Condition

For $\beta > 0$ let $\mathcal{T}(B) := \limsup_{r \to 0^+} \phi^+_{r,B}$.

Definition

(Time regularity, general case) For $\beta(x)$ the local time exponent, we say X satisfies $E(\beta)$ if β is bounded and for all $x \in X, 0 < r < \frac{\mathrm{diam}(X)}{2}$, we have

$$\mathcal{T}(B_r(x)) \approx r^{\beta(x)}.$$

[a]

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[a] [7] considers a similar condition under assumption that a heat kernel exists.

Example

Let B the open ball of radius R about the origin in $\mathbb{R}^n$ under the euclidean norm $|\cdot|$. If $B_r(x) \subset B$ we have $\phi_r(x) = \left(\frac{n+2}{n}\right)(R^2 - |x|^2)$. Hence $\mathcal{T}(B_r(x)) = (\frac{n+2}{n})r^2$.

Dirichlet spectrum lower bound

Let $B = B_R(x_0)$ and assume $\mu((B^c)^\circ) > 0$. Then define $\mathcal{L}_r^B$ on $L^2(B, \nu_r)$ by $\mathcal{L}_r^B f(x) = \chi_B(x)\mathcal{L}_r(\chi_B f)(x)$ where we define $\chi_B f$ outside B to be 0. We have the exit time equation $\mathcal{L}_r^B \phi_{B,r}(x) = 1$.[5] Then one can show $\mathcal{L}_r^B$ has a ν_r-symmetric Green's function $g^B(x, y)$ such that

$$\int g^B(x, y) d\nu_r(y) = \phi_{B,r}(x).$$

Theorem

If λ_r^B is the bottom of the spectrum of $\mathcal{L}_r^B$ on $L^2(B, \nu_r)$ then

$$\lambda_r^B \geq \frac{c}{R^{\beta(x_0)}}$$

for some constant $c > 0$ independent of r, R, and x_0.

[5] Compare to "torsion function" and "torsional rigidity" on a Riemannian manifold.

$\Gamma-$Convergence[6]

If $\mathcal{X}$ is a separable second countable topological space with neighborhood basis at x $\mathcal{B}(x)$ then
$(\Gamma\text{-}\liminf_{n\to\infty} f_n)(x) := \sup_{U\in\mathcal{B}(x)} \liminf_{n\to\infty} \inf_{y\in U} f_n(y)$ and
$(\Gamma\text{-}\limsup_{n\to\infty} f_n)(x) := \sup_{U\in\mathcal{B}(x)} \limsup_{n\to\infty} \inf_{y\in U} f_n(y)$. We say f_n $\Gamma-$converges to f if the two are equal.

Theorem

[a] Every sequence of non-negative Markovian forms forms on a separable Hilbert space $L^2(X,\mu)$ has a subsequence $\Gamma-$converging to a closed, non-negative, Markovian form (i.e. a Dirichlet form).

[a]proven in Mosco *Composite media and asymptotic Dirichlet forms*[4].

This Theorem was applied in the Kumagai-Sturm construction in [1]. In that paper $\Gamma-$limits of approximate Dirichlet forms
$\mathcal{E}_r(u) = \int_X \int_X |u(x) - u(y)|^2 k_r(x,y) d\mu(x) d\mu(y)$ are considered.
Particular attention is paid to the choice $k_r(x,y) = \frac{1}{h(r)\mu(B_r(x))} \chi_{B_r(x)}$.

[6]For more information see book by Dal Maso [5]

In our case the process $(X_t^{(r)})_{t\geq 0}$ leads us to consider approximate forms given by
$\mathcal{E}_r(f) = \langle f, \mathcal{L}_r f\rangle_{L^2(X,\nu_r)} = \frac{1}{Z_r}\int_X \int_{B_r(x)} |f(y) - f(x)|^2 d\mu(y) d\mu(x)$, where
$Z_r = \langle v_r r^{\beta}\rangle_{\mu} = \int_X \mu(B_r(x)) r^{\beta(x)} d\mu(x)$, $d\nu(x) = \mu(B_r(x)) r^{\beta(x)} d\mu(x)$.

Theorem

A sequence $(r_n)_{n=1}^{\infty}$ of positive numbers decreasing to 0 can be chosen such that $\mathcal{E} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{E}_{r_n}$ exists, and $\mathcal{D}(\mathcal{E})$ contains an algebra of bounded measurable functions separating points[a].

[a]this algebra is generated by pointwise limits of exit time functions from balls in a neighborhood basis.

[1]Takashi Kumagai, Karl-Theodor Sturm, *Construction of diffusion processes on fractals, d-sets, and general metric measure spaces*(2005).

[2]András Telcs, *The art of random walks*, 2006.

[3]Jean Bellissard, *Diffusion on compact metric spaces*, Unpublished private notes.

[4]Umberto Mosco, *Composite media and asymptotic Dirichlet forms*, 1994.

[5]Gianni Dal Maso, *An introduction to* Γ*-convergence*, 2012.

[6]J. Dever, *Local Hausdorff measure*, ArXiv (2016).

[7]Alexander Grigor'yan, *Heat kernels on manifolds, graphs and fractals*, 2001.

[8]Harjulehto, Hästö, and Latvala, *Sobolev embeddings in metric measure spaces with variable dimension*(2006).

Thank you.