A family of self-avoiding walks on the Sierpiński gasket

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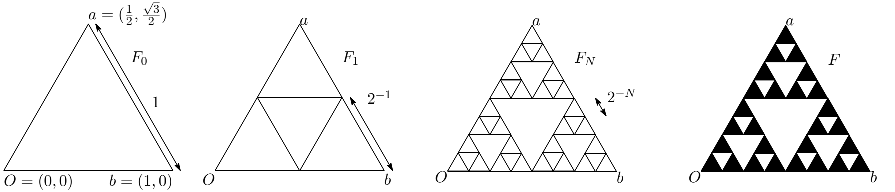
Outline

1. Pre-Sierpinski gasket
2. A set of probability measures with multi parameter
3. Scaling limit
4. Self-avoiding property

1. Pre-Sierpiński gasket



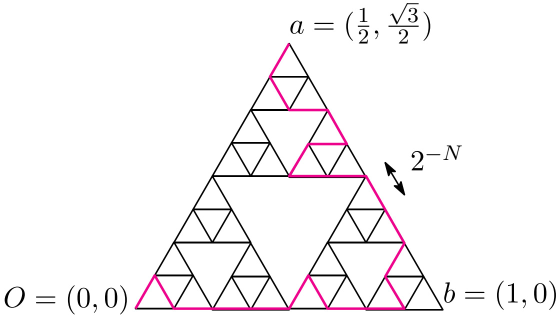
Let F_0 be a triangle $\triangle Oab$,
F_N be the graph with edges of length 2^{-N}, and
$F = c\ell(\cup_{N=0}^{\infty} F_N)$ be a (finite) Sierpiński gasket.

- Next, define self-avoiding paths on F_N and probability measures on the path spaces inductively.

Self-avoiding paths

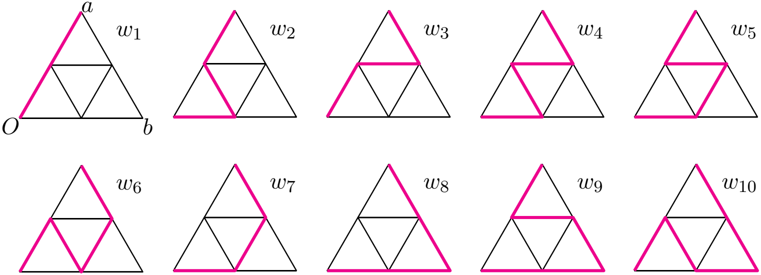
w: self-avoidng path on F_N if

$w(0) = O$, $(w(i), w(i+1)) \in \{$ edges on $F_N\}$, $w(i) \in \{$ vertices on $F_N\}$, $w(i) \neq w(j)$ $(i \neq j)$, and $w(\ell(w)) = a$.



2. A set of probability measures with multi parameter

Self-avoiding paths on F_1:

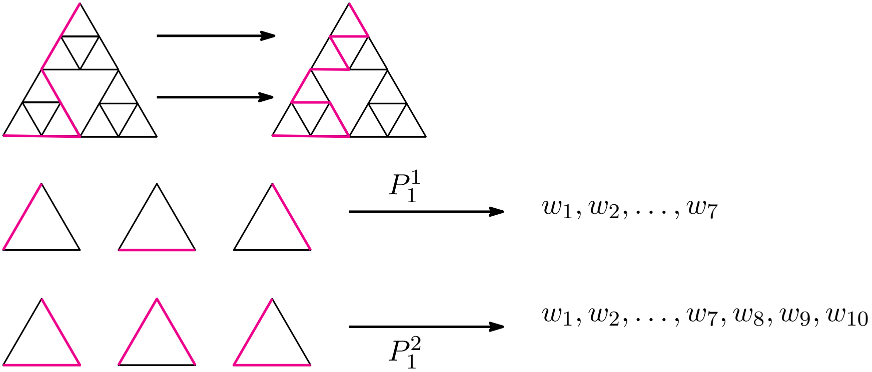


Given $p_i, q_i \geq 0$, $\sum_{i=1}^{10} p_i = 1$, $\sum_{i=1}^{10} q_i = 1$, $p_8, p_9, p_{10} = 0$, define

$$P_1^1[w_i] = p_i, \quad P_1^2[w_i] = q_i.$$

Branching

Idea: P^1_{N+1} and P^2_{N+1} are obtained by the following branching.



Following this idea, define $X_N(w)(j) = w(j)$, $j = 0, 1, \ldots, \ell(w)$, where w is a self-avoiding path on F_N.

(Suppose that the first branching follows P^1_1.)

Example

Loop-erased random walk (defined by erasing loops in descending order of the size of loops from a simple random walk) on F_N (Hattori, Mizuno 14') is included as a special case.

$$p_1 = \frac{1}{2},\ p_2 = p_3 = p_7 = \frac{2}{15},\ p_4 = p_5 = p_6 = \frac{1}{30},\ p_8 = p_9 = p_{10} = 0,$$

$$q_1 = \frac{1}{9},\ q_2 = q_3 = q_7 = \frac{11}{90},\ q_4 = q_5 = q_6 = \frac{2}{45}, q_7 = \frac{8}{45}, q_8 = \frac{2}{9},$$
$$q_9 = q_{10} = \frac{1}{18}.$$

Additionally, 'standard' self-avoiding walk (HHK 91'), loop-erased self-repelling walk (HOO, 17') are included as special cases respectively (we omit details here).

Assigning different values to p_i, q_i gives different SAWs.

3. Scaling limit

Let $\mathbf{p} = (p_1, \ldots, p_{10}, q_1, \ldots, q_{10})$ and $X_N(w)(t) = w(t)$, $t \in [0, \infty)$: a self avoiding walk on F_N (linear interpolated).

Theorem (scaling limit)

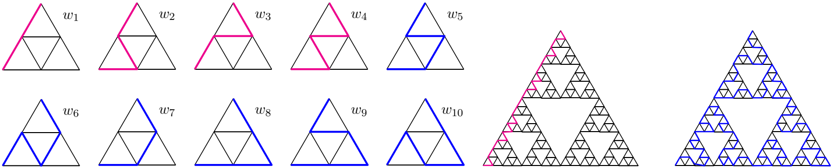
For any $\mathbf{p}$, there exists $\lambda = \lambda(\mathbf{p})(2 \leq \lambda \leq 3)$ such that

$$X_N(\lambda^N t) \to X(t) \quad a.s. \quad \text{as} \quad N \to \infty.$$

and $d_H = \log \lambda / \log 2$(Hausdorff dimension of the path) with probability 1.

- d_H takes any values from 1 to $\log 3 / \log 2$.
- X_N is self-avoiding, but $X = X(\mathbf{p})$ is not neccesarily self-avoiding and the number of triangles produced at each branching affects the self-avoiding property.

Two extream cases

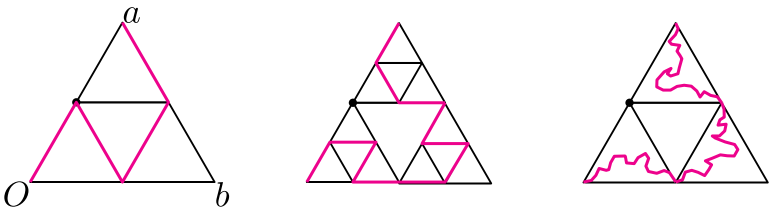


- If $p_1 + p_2 + p_3 + p_4 = q_1 + q_2 + q_3 + q_4 = 1$, then, with probability 1, $d_H = 1$ and X is an uniform linear motion.
- If $p_5 + p_6 + p_7 = q_5 + \cdots + q_{10} = 1$, then, with probability 1, $d_H = \log 3 / \log 2$, that is, X fills the state space(the SG), and the speed of the motion is constunt. We call it Peano curve.

4. Self-avoiding property

Theorem (Self-avoiding)

If $p_5 + p_6 + p_7 < 1, q_5 + \cdots + q_{10} < 1$, then the scaling limit X is self-avoiding.



Asymptotic behavior

Suppose that $p_5 + p_6 + p_7 < 1, q_5 + \cdots + q_{10} < 1$. In this case, we have some results about asymptotic behavior.

Theorem (Short time behavior)

Let $\gamma = \log 2 / \log \lambda$. There exist positive constants C_1, C_2 such that for all $s > 0$

$$C_1 \leq \lim_{t \to 0} \frac{E[|X(t)|^s]}{t^{\gamma s}} \leq C_2.$$

Theorem (Laws of the iterated logarithm)

There exist positive constants C_3, C_4 such that

$$C_3 \leq \limsup_{t \to 0} \frac{|X(t)|}{t^{\gamma} (\log \log (1/t))^{1-\gamma}} \leq C_4 \quad a.s.$$

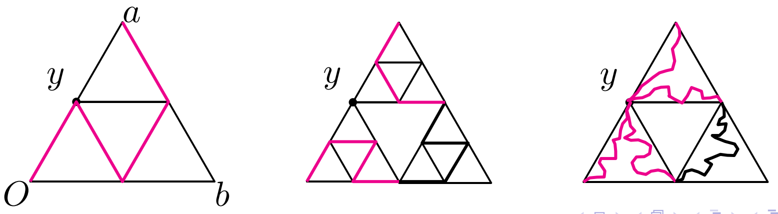
Self-intersections

Theorem

If $p_5 + p_6 + p_7 = 1$, then, with probability 1, the scaling limit X has infinitely many self-intersections.

If $p_7 < 1$ and $q_5 + \cdots + q_{10} < 1$, then $d_H < \log 3 / \log 2$.

Example:
If the branching is $w_6 \to (w_5, w_6, w_7) \to \dots$, then following every branchings in the sets of triangles including y follows P_1^1.
Iterating this, therefore, X reaches y from opposite side of x.



Conclusion

- Using the branching method, we have studied a family of self-avoiding walks on the SG.
- The limit is divided into four types in the meaning of shape.
- The case that the limit have intersections but does not fill the state space is not included in previous models.
- For the self-avoiding case, we have studied asymptotic behavior.

References

[1]Hattori, K., Mizuno, M. : *Loop-erased random walk on the Sierpinski gasket*, Stoch. Proc. Applic, **124**, (2014) 566–585.

[2]Hattori, K., Ogo, N., Otsuka, T. : *A family of self-avoiding random walks interpolating the loop-erased random walk and a self-avoiding walk on the Sierpinski gasket*, Discrete Contin. Dyn. Syst. S, **10**, (2017) 289–312.