Boundary value problems for harmonic functions on domains in Sierpinski gaskets

Hua Qiu

(Joint work with Shiping Cao)

Nanjing University

Email: huaqiu@nju.edu.cn

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Dirichlet problem

For a domain Ω, finding a function which is *harmonic* in Ω that takes prescribed values on $\partial\Omega$,

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f, f \in C(\partial\Omega). \end{cases} \tag{1}$$

Backgrounds

Dirichlet problem

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For a bounded domain Ω with *sufficiently smooth boundary* $\partial\Omega$, the problem is always *solvable*,

$$u(x) = \int_{\partial\Omega} f(s)\partial_n G(x,s)ds, \quad \forall x \in \Omega.$$

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$G(x,y)$ is the *Green's function*, $\partial_n G(x,s)$ is the *Poisson kernel* for Ω.

Harmonic extension algorithm *(w.r.t. the standard energy)*

- Sierpinski gasket $\mathcal{SG}$ the "$\frac{1}{5}-\frac{2}{5}$" rule
- $\mathcal{SG}_3$ the "$\frac{8}{15}-\frac{4}{15}-\frac{3}{15}$" rule
- $\mathcal{SG}_l$, $l \geq 2$, $\mathcal{SG}_l = \bigcup_{i=0}^{\frac{l^2+l-2}{2}} F_i \mathcal{SG}_l$

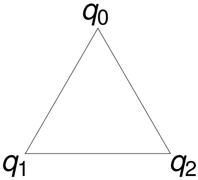
  $V_0 = \{q_0, q_1, q_2\}$, $V_m = \bigcup_{i=0}^{\frac{l^2+l-2}{2}} F_i V_{m-1}$

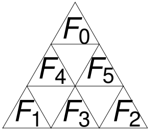
  Solving linear equations

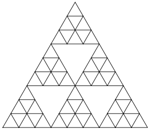
  $\sum_{y\sim_1 x}(h(y)-h(x)) = 0, \forall x \in V_1 \setminus V_0$

  harmonic extension matrices A_i's: $h \circ F_i|_{V_0} = A_i h|_{V_0}$

  local algorithm: $h \circ F_{wi}|_{V_0} = A_i h \circ F_w|_{V_0}$
- p.c.f. self-similar set (omit)







The problem

Consider Dirichlet problems on *connected subsets* of p.c.f. self-similar sets (*with fractal boundary*)
we will focus on subdomains in $\mathcal{SG}_l$, $l \geq 2$

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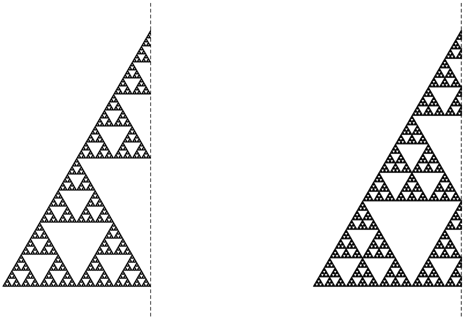
we will focus on subdomains in $\mathcal{SG}_l$, $l \geq 2$

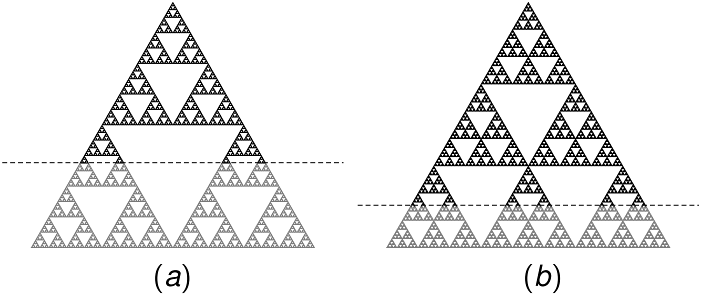
- half domains



(a) (b)

Half domains of $\mathcal{SG}$ and $\mathcal{SG}_3$.

- upper domains (lower domains)



Upper and lower domains in $\mathcal{SG}$ and $\mathcal{SG}_3$.

The problem

Proposition 1.

Let Ω be a half, upper or lower domain in $\mathcal{SG}_l$, $l \geq 2$. The Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f, f \in C(\partial\Omega) \end{cases}$$

has a unique solution.

Our goal:

The problem

Proposition 1.

Let Ω be a half, upper or lower domain in $\mathcal{SG}_l$, $l \geq 2$. The Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f, f \in C(\partial\Omega) \end{cases}$$

has a unique solution.

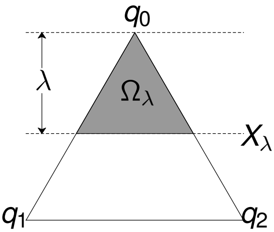
Our goal:

- *to find the exact harmonic extension algorithm in Ω*
- *to estimate the energy of harmonic solutions in terms of boundary values*

Prior works

Upper domains in $\mathcal{SG}$ (Strichartz, 1999; Owen and Strichartz, 2012; Guo, Kogan, Q. and Strichartz, 2014)

- Extension algorithm
- Energy estimate



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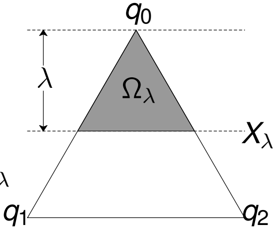


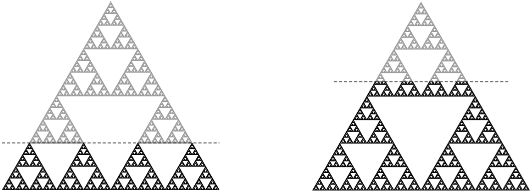
Main tool: Haar series expansion along X_λ
Haar basis $\{\phi_w\} \cup \{1\}$ for $L^2(X_\lambda, \mu)$
the standard measure μ on X_λ
The only generator of the Haar basis is antisymmetric, its extended harmonic solution is also antisymmetric, so can be localized to any small scales along the boundary X_λ.
Need symmetry, could not be extended to $\mathcal{SG}_l$, $l \geq 3$

Prior works

Lower domains in $\mathcal{SG}$ (Strichartz, 1999)

very special case, the domains are made up of 2^m adjacent triangles of size 2^{-m} lying on the bottom line of $\mathcal{SG}$

- *the boundary is a finite set*
- *the energy estimate is unknown*



Two typical domain Ω_λ^-'s, λ is (or not) a dyadic rational.

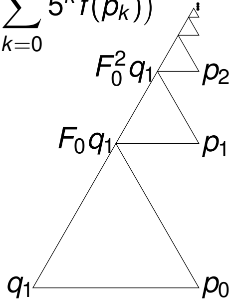
For general lower domains, we have little knowledge.

Prior works

Half domain of $\mathcal{SG}$ (Li and Strichartz, 2014)

- Extension algorithm

$$u(F_0^m q_1) = \frac{1}{5^m}\Big(f(q_1) - \frac{3}{7}\sum_{k=0}^{\infty}(\frac{1}{3})^k f(p_k) + \frac{10}{7}\sum_{k=0}^{\infty} 5^k f(p_k)\Big) + \frac{3}{7}\sum_{k=0}^{\infty}(\frac{1}{3})^k f(p_{m+k})$$

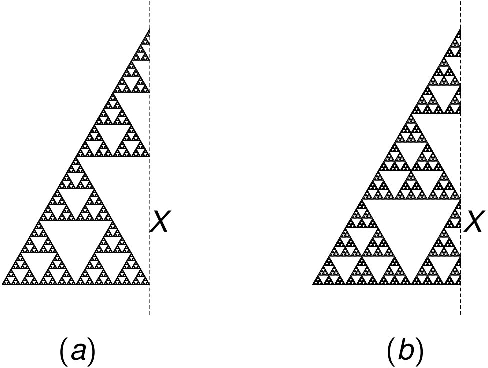


- Energy estimate

$$\mathcal{E}(u) \asymp (f(q_1) - f(p_0))^2 + \sum_{k=0}^{\infty}(\frac{5}{3})^k (f(p_k) - f(p_{k+1}))^2$$

- Method: solving systems of infinite linear equations, multiplying infinite matrices

Prior works



Half domains of $\mathcal{SG}$ and $\mathcal{SG}_3$.

For $\mathcal{SG}$, X consists a countable infinite set
For $\mathcal{SG}_l$, $l \geq 3$, X is a Cantor set
The approach is not applicable for $\mathcal{SG}_l$, $l \geq 3$

New method

- There exist *crucial points* for certain domains. We only need to solve finite linear equations.
- The normal derivative of the harmonic solution at a boundary point is determined by the integral of boundary data w.r.t. a signed measure.

Extension 1: Half domain of $\mathcal{SG}_3$

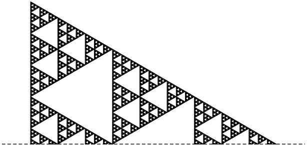
- Structure of half domain

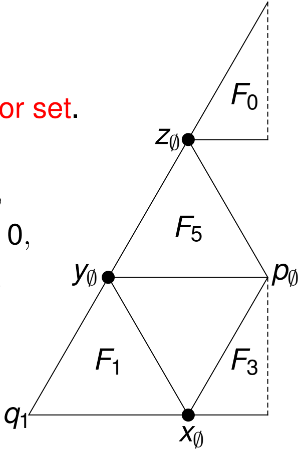
  $\bar\Omega = F_1\mathcal{SG}_3 \cup F_5\mathcal{SG}_3 \cup F_0\bar\Omega \cup F_3\bar\Omega.$

  The boundary set $X = \partial\Omega \setminus \{q_1\}$ is a Cantor set.

- Basic equations

$$\begin{cases} \frac{7}{15}\partial_n^{\leftarrow} u(x_\emptyset) + 2u(x_\emptyset) - u(y_\emptyset) - f(q_1) = 0, \\ 4u(y_\emptyset) - u(x_\emptyset) - u(z_\emptyset) - f(q_1) - f(p_\emptyset) = 0, \\ \frac{7}{15}\partial_n^{\leftarrow} u(z_\emptyset) + 2u(z_\emptyset) - u(y_\emptyset) - f(p_\emptyset) = 0. \end{cases}$$





Extension 1: Half domain of $\mathcal{SG}_3$

- Calculation of $\partial_n^{\leftarrow} u(x_\emptyset)$ and $\partial_n^{\leftarrow} u(z_\emptyset)$ in terms of boundary data

  Observation 1.

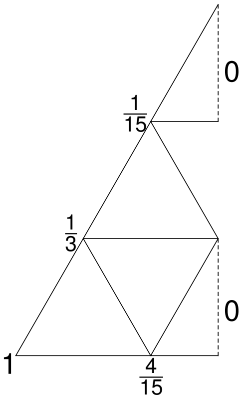
  $$\partial_n^{\leftarrow} u(x_\emptyset) = \tfrac{15}{7}\partial_n^{\leftarrow}(u\circ F_3)(q_1),$$
  $$\partial_n^{\leftarrow} u(z_\emptyset) = \tfrac{15}{7}\partial_n^{\leftarrow}(u\circ F_0)(q_1).$$

  Observation 2.—-Gauss-Green's formula

  Roughly, we have

  $$\mathcal{E}_\Omega(u,h_a) = \partial_n^{\leftarrow} u(q_1) = \int_{\partial\Omega} u(x)\partial_n h_a(x)d\mu(x),$$

  h_a being the *restriction* of a antisymmetric harmonic function on Ω.

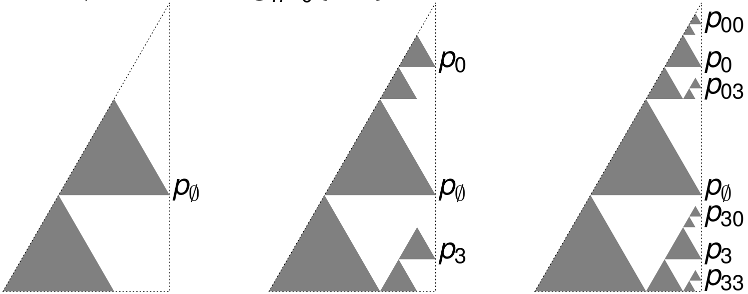


Extension 1: Half domain of $\mathcal{SG}_3$

By using a sequence of *simple sets* to approximate $\bar{\Omega}$, we prove

$$\partial_n^{\leftarrow} u(q_1) = 3f(q_1) + \sum_{w \in \tilde{W}_*} \partial_n^{\rightarrow} h_a(p_w) f(p_w).$$

$p_w = F_w p_\emptyset, w \in \tilde{W}_* = \bigcup_{n=0}^{\infty} \{0,3\}^n.$



Extension 1: Half domain of $\mathcal{SG}_3$

Introduce the (probability) measure

$$\mu_a = \frac{1}{3} \sum_{w \in \tilde{W}_*} (-\partial_n^{\rightarrow} h_a(p_w)) \delta_{p_w},$$

then

$$\partial_n^{\leftarrow} u(q_1) = 3f(q_1) - 3 \int_X f d\mu_a (= \mathcal{E}_\Omega(u, h_a) \text{ if } u \in dom\mathcal{E}_\Omega).$$

For this reason, we may view the measure $3\delta_{q_1} - 3\mu_a$ as the normal derivative of h_a along $\partial\Omega$.

Exact data: $-\partial_n^{\rightarrow} h_a(p_w) = \frac{6}{7} \prod_{i=1}^{|w|} \mu_{w_i}$, for $\mu_0 = \frac{1}{7}, \mu_3 = \frac{4}{7}$.

Extension 1: Half domain of $\mathcal{SG}_3$

Theorem 1.1. Extension algorithm

There exists a unique solution of the Dirichlet problem (1) on the half domain of SG_3. In addition, we have

$$u(x_\emptyset) = \frac{4}{15}f(q_1) + \frac{1}{15}f(p_\emptyset) + \frac{1}{30}\int_X f\circ F_0 d\mu_a + \frac{19}{30}\int_X f\circ F_3 d\mu_a,$$

$$u(y_\emptyset) = \frac{1}{3}f(q_1) + \frac{1}{3}f(p_\emptyset) + \frac{1}{6}\int_X f\circ F_0 d\mu_a + \frac{1}{6}\int_X f\circ F_3 d\mu_a,$$

$$u(z_\emptyset) = \frac{1}{15}f(q_1) + \frac{4}{15}f(p_\emptyset) + \frac{19}{30}\int_X f\circ F_0 d\mu_a + \frac{1}{30}\int_X f\circ F_3 d\mu_a.$$

Extension 1: Half domain of $\mathcal{SG}_3$

Theorem 1.2. Energy estimate

We have the energy estimate that

$$C_1 Q(f) \le \mathcal{E}(u) \le C_2 Q(f),$$

where

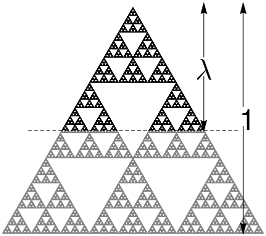
$$Q(f) = \big(f(q_1) - f(p_\emptyset)\big)^2 + \sum_{w \in \tilde{W}_*} \left(\frac{15}{7}\right)^{|w|} \Big(\big(f(p_w) - f(p_{w0})\big)^2 + \big(f(p_w) - f(p_{w3})\big)^2 \Big).$$

Remark 1. Theorem 1.2 is motivated by Li and Strichartz, 2014.
Remark 2. Both Theorem 1.1 and 1.2 can be extended to $\mathcal{SG}_l$ cases.

Extension 2: Upper domains of $\mathcal{SG}_3$

- Notations. Assume $q_0 = (\frac{1}{\sqrt{3}}, 1),\ q_1 = (0,0),\ q_2 = (\frac{2}{\sqrt{3}}, 0)$.
  For $0 < \lambda \leq 1$, define the *upper domain*

$$\Omega_\lambda = \{(x,y) \in \mathcal{SG}_3 \setminus V_0 | y > 1 - \lambda\},$$



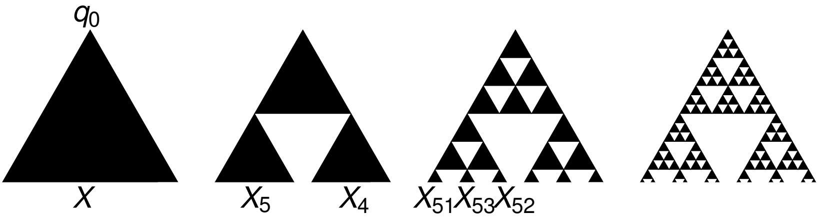
Extension 2: Upper domains of $\mathcal{SG}_3$

Write $\lambda = \sum_{k=1}^{\infty} \iota_k 3^{-m_k}$, with $0 < m_1 < m_2 < \cdots$ and $\iota_k = 1$ or 2. The boundary is $\partial\Omega = \{q_0\} \bigcup X_\lambda$, with X_λ *homeomorphic* to

$$\Sigma_\lambda = \prod_{k=1}^{\infty} S_{\iota_k},$$

where $S_1 = \{4, 5\}$, $S_2 = \{1, 2, 3\}$.

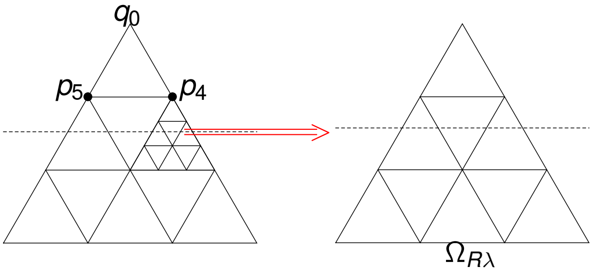
Denote the *cylinder* of X corresponding to $w \in W_m^\lambda = \prod_{k=1}^{m} S_{\iota_k}$ by X_w.



- Basic equations

  Case 1. $(\iota_1 = 1)$ $\lambda = (1 + R\lambda)3^{-m_1}$, with $0 < R\lambda \leq 1$

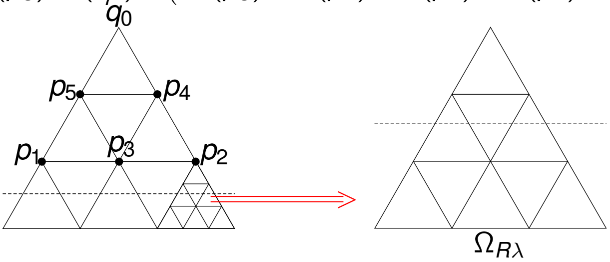
$$\begin{cases} \partial_n^{\uparrow} u(p_5) + (\frac{15}{7})^{m_1} \big(2u(p_5) - u(p_4) - f(q_0)\big) = 0, \\ \partial_n^{\uparrow} u(p_4) + (\frac{15}{7})^{m_1} \big(2u(p_4) - u(p_5) - f(q_0)\big) = 0. \end{cases}$$



Extension 2: Upper domains of $\mathcal{SG}_3$

Case 2. $(\iota_1 = 2)$ $\lambda = (2 + R\lambda) \cdot 3^{-m_1}$, with $0 < R\lambda \leq 1$

$$\begin{cases} 4u(p_5) - u(p_1) - u(p_3) - u(p_4) - f(q_0) = 0, \\ 4u(p_4) - u(p_2) - u(p_3) - u(p_5) - f(q_0) = 0, \\ \partial_n^{\uparrow} u(p_1) + (\frac{15}{7})^{m_1} (2u(p_1) - u(p_3) - u(p_5)) = 0, \\ \partial_n^{\uparrow} u(p_2) + (\frac{15}{7})^{m_1} (2u(p_2) - u(p_3) - u(p_4)) = 0, \\ \partial_n^{\uparrow} u(p_3) + (\frac{15}{7})^{m_1} (4u(p_3) - u(p_1) - u(p_2) - u(p_4) - u(p_5)) = 0. \end{cases}$$



- Calculation of Normal derivatives

  Still the Gauss-Green's formula

$$\partial_n^{\uparrow} u(q_0) = \partial_n^{\uparrow} h_0(q_0) f(q_0) - \int_{X_\lambda} f d\mu_\lambda (= \mathcal{E}_{\Omega_\lambda}(u, h_0) \text{ if } u \in dom\mathcal{E}_{\Omega_\lambda}). \tag{2}$$

  h_0 is harmonic in Ω_λ with $h_0(q_0) = 1$ and $h_0 \equiv 0$ on X_λ.

  Observation 1. $-\mu_\lambda$ is the "*normal derivative*" of h_0 on X_λ

  Observation 2. $\partial_n^{\uparrow} h_0(p_i) = (\frac{15}{7})^{m_1} h_0(p_i) \partial_n^{\uparrow} h_0^{R\lambda}(q_0)$

  Calculate μ_λ on each cylinder X_w *inductively* using Observation 2.

  The proof for formula (2) involves a series of *simple sets* approximating to the domain.

Extension 2: Upper domains of $\mathcal{SG}_3$

The extension algorithm is derived by solving the basic equations.

Theorem 2.1. Extension algorithm

There exists a unique solution of the Dirichlet problem (1) on the upper domain Ω_λ. In addition, we have explicit extension algorithm for $u(p_i)'s$ in terms of $f(q_0)$ and integrals of boundary value data w.r.t. $\mu_{R\lambda}$.

- Haar series

Definition:

If $\iota_{n+1} = 1$, define $\psi_w^{(1)} = \chi_{X_{w4}} - \chi_{X_{w5}}$, for $w \in W_n^\lambda$.

If $\iota_{n+1} = 2$, define

$$\begin{cases} \psi_w^{(1)} = \chi_{X_{w1}} - \chi_{X_{w2}} \\ \psi_w^{(2)} = \mu_w^{-1}(\mu_{w3}\chi_{X_{w1}} - 2\mu_{w1}\chi_{X_{w2}} + \mu_{w3}\chi_{X_{w2}}) \end{cases}, \text{ for } w \in W_n^\lambda.$$

$\psi_\emptyset^{(1)}$: 1 $\quad -1$

$\psi_\emptyset^{(1)}$: 1 $\quad 0$ $\quad -1$

$\psi_\emptyset^{(2)}$: μ_3 $\quad -2\mu_1$ $\quad \mu_3 = \mu_2$

Properties:

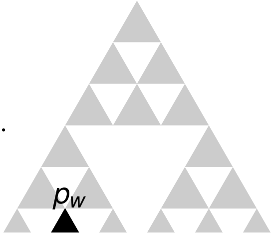
1. $\{\psi_w^{(j)}\}_{w \in W_*^\lambda} \bigcup \{1\}$ form an orthogonal basis in $L^2(X_\lambda, \mu_\lambda)$.

2. $\|\psi_w^{(j)}\|_{L^2(X_\lambda, \mu_\lambda)} \asymp \mu_w^{\frac{1}{2}}$

Extension 2: Upper domains of $\mathcal{SG}_3$

Let $h_w^{(j)}$ be the harmonic functions with

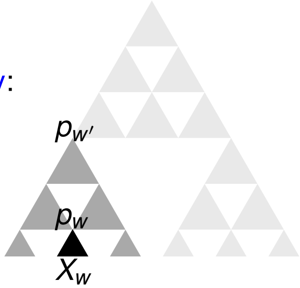
$$h_w^{(j)}|_{X_\lambda} = \psi_w^{(j)}, \quad h_w^{(j)}(q_0) = 0.$$



Then we have

1. $h_w^{(j)}$ supports locally, $\partial_n^{\uparrow} h_w^{(j)}(p_w) = 0$
2. $\mathcal{E}_\Omega(h_w^{(j)}) \asymp (\frac{15}{7})^{m_{|w|+1}}$.
3. $\{h_w^{(j)}\} \cup \{h_0\}$ are pairwise orthogonal in energy.

Extension 2: Upper domains of $\mathcal{SG}_3$



A *brief explanation* for the orthogonality in energy:

Case 1. $X_w \bigcap X_{w'} = \emptyset$: Disjoint support.

Case 2. $w = w', j \neq j'$: Different symmetry.

Case 3. $X_w \subset X_{w'}$:

$$\begin{aligned}\mathcal{E}_{\Omega_\lambda}(h_w^{(j)}, h_{w'}^{(j')}) &= \mathcal{E}_{\mathrm{supp} h_w^{(j)}}(h_w^{(j)}, h_{w'}^{(j')}) \\ &= \partial_n^{\uparrow} h_w^{(j)}(p_w) h_{w'}^{(j')}(p_w) + \int_{X_w} h_{w'}^{(j')} \partial_n h_w^{(j)} \\ &= 0 + \int_{X_w} C \partial_n h_w^{(j)} = 0\end{aligned}$$

(The total of normal derivative on X_w is zero.)

Extension 2: Upper domains of $\mathcal{SG}_3$

Theorem 2.2. Energy Estimate

Let u be the harmonic function in Ω_λ with boundary values $u(q_0) = a$ and $u|_{X_\lambda} = f$, where

$$f = b + \sum_{w \in W_*^\lambda} \sum_{j \leq \iota_{|w|+1}} c_w^{(j)} \psi_w^{(j)}.$$

Then $\mathcal{E}_{\Omega_\lambda}(u)$ is bounded above and below by a multiple of

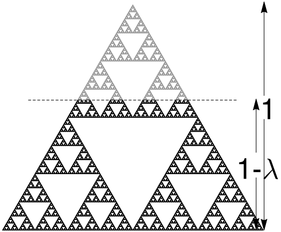
$$(\frac{15}{7})^{m_1}(a-b)^2 + \sum_{n=0}^{\infty} \sum_{w \in W_n^\lambda} \sum_{j=1}^{\iota_{n+1}} (\frac{15}{7})^{m_{n+1}} |c_w^{(j)}|^2.$$

Remark. Both Theorem 2.1 and 2.2 can be extended to $\mathcal{SG}_l$ cases.

Extension 3: Lower domains of $\mathcal{SG}$

- Notations. Assume $q_0 = (\frac{1}{\sqrt{3}}, 1),\ q_1 = (0,0),\ q_2 = (\frac{2}{\sqrt{3}}, 0)$.
  For $0 \leq \lambda < 1$, define the lower domain

$$\Omega_\lambda^- = \{(x,y) \in \mathcal{SG} \setminus V_0 | y < 1 - \lambda\}$$



Consider the binary expansion

$$\lambda = 0.e_1 e_2 \cdots = \sum_{k=1}^{\infty} e_k(\lambda) 2^{-k}.$$

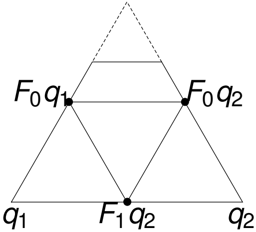
Define $S\lambda = 0.e_2 e_3 \cdots$.

Extension 3: Lower domains of $\mathcal{SG}$

- Basic equations

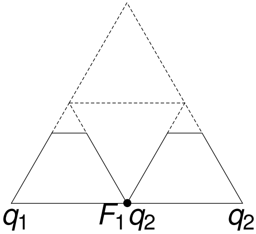
Case 1. $e_1 = 0$

$$\begin{cases} 4u(F_1q_2) - f(q_1) - f(q_2) - u(F_0q_1) - u(F_0q_2) = 0, \\ \frac{3}{5}\partial_n^{\leftarrow} u(F_0q_1) + \big(2u(F_0q_1) - u(F_1q_2) - f(q_1)\big) = 0, \\ \frac{3}{5}\partial_n^{\rightarrow} u(F_0q_2) + \big(2u(F_0q_2) - u(F_1q_2) - f(q_2)\big) = 0. \end{cases}$$



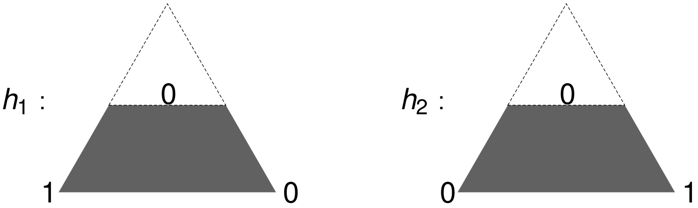
Case 2. $e_1 = 1$

$$\partial_n^{\rightarrow} u(F_1q_2) + \partial_n^{\leftarrow} u(F_1q_2) = 0.$$



Extension 3: Lower domains of $\mathcal{SG}$

- Basic ideas
  Introduce two harmonic functions h_1, h_2.



  Calculate $\partial_n h_1, \partial_n h_2$ on the boundary.
  Use Gauss-Green's formula to get the normal derivatives, and solve the basic equations.
- We have the extension algorithm (omit), could be extended to $\mathcal{SG}_l$ case.
  However, the energy estimate remains open.

Related problems

- Dirichlet to Neumann map
- trace problem and extension problem
- other Sobolev types of norms
- Green's function
- spectral decimation and spectral analysis
- other subdomains in p.c.f. self-similar sets

Thank you!