A counterexample to the "hot spots" conjecture on nested fractals

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(With K.-S. Lau and X.-H. Li)

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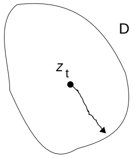
Motivation

- The "hot spots" conjecture was posed by Rauch in 1974.
- D: *open connected bounded subset of* $\mathbb{R}^d$.
- $u(t,x), t \geq 0, x \in D$: *the solution of*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta_x u(t,x), & x \in D,\ t > 0, \\ \frac{\partial u}{\partial n}(t,x) = 0, & x \in \partial D,\ t > 0, \\ u(0,x) = u_0(x), & x \in D. \end{cases}$$



- Informally speaking: Suppose that

$$u(z_t, t) = \max\{u(x,t) : t \in \overline{D}\}.$$

Then, we conjecture that for "most" initial conditions,

$$\lim_{t\to\infty} d(z_t, \partial D) = 0.$$

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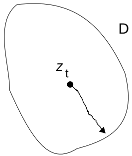

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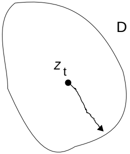

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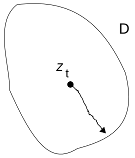

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- Let $0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots$ be the spectrum of $\Delta_N(D)$.
- $\mathcal{N}_2$: the set of all N-eigenfunctions corresponding to μ_2.
- "Typically", $\exists a_1 \in \mathbb{R}$ and $\varphi_2 \in \mathcal{N}_2$ with $\varphi_2 \neq 0$, s.t.

$$u(t,x) = a_1 + \varphi_2(x)e^{-\mu_2 t} + R(t,x),$$

  where $R(t,x)$ goes to 0 faster than $e^{-\mu_2 t}$, as $t \to \infty$.
- (HSC) $\forall \varphi_2 \in \mathcal{N}_2$ which is not identically 0, φ_2 attains its maximum and minimum on ∂D (only).

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Motivation

(HSC) holds for following domains:

- (Well known) balls, annulus;
- (Kawohl, LNM, 1985)
  $D = D_1 \times (0, a)$, where $\partial D_1 \in C^{0,1}$;
- (Bañuelos-Burdzy, JFA, 1999; Pascu, TAMS, 2002)
  convex domain which has a line of symmetry;
- (Ata-Burdzy, JAMS, 2004)
  lip domains: bounded Lipschitz planar domain

$$D = \{(x, y) : f_1(x) < y < f_2(x)\},$$

  where f_1, f_2: Lipschitz functions with Lipschitz constant 1;
- (Miyamoto, J Math Phy, 2009)
  convex planar domains D with $\mathrm{diam}(D)^2/Area(D) < 1.378$.

Motivation

The hot spots conjecture fails for some planar domains:

- Burdzy-Werner (Ann Math, 1999): a bounded connected planar domain D (with two holes) s.t. $multi(\lambda_2) = 1$, and φ_2 attains its strict maximum at an interior point of D.
- Bass-Burdzy (Duke Math J, 2000): a bounded Lipschitz planar domain D s.t. $multi(\lambda_2) = 1$, and φ_2 attains its strict maximum and strict minimum at interior points of D.
- Burdzy (Duke Math J, 2005): a domain with one hole with above property.

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- *Does (HSC) hold for all bounded convex domains $D \subset \mathbb{R}^d$?*
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Question

How about p.c.f. self-similar sets?

Hot spots conjecture on p.c.f. self-similar sets

By using the spectral decimation method, we know that (HSC) holds on:

- Sierpinski gasket (R, Nonl Anal, 2012);
- Level-3 SG (R-Zheng, Nonl Anal, 2013);
- Higher dimensional SG (Li-R, CPAA, 2016);

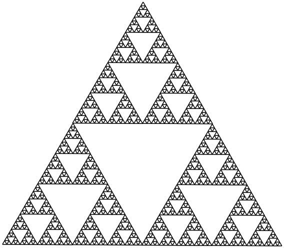

Figure: SG

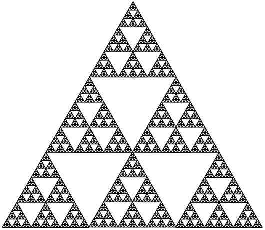

Figure: SG_3

Question

- Does (HSC) hold on all p.c.f. self-similar sets introduced by Kigami?
- Does (HSC) hold on all nested fractals introduced by Linstrøm?

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Counterexample:

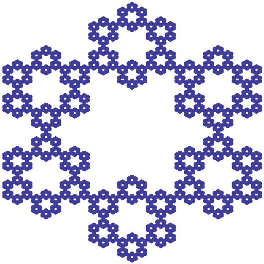

Figure: Hexagasket (HG)

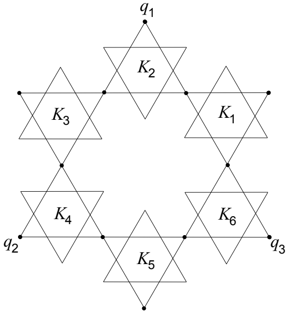

Figure: V_0 and V_1

- $F_k(x) = \frac{x}{3}e^{ik\pi/3} + \frac{2}{3}p_k, \;\; k = 1, \ldots, 6.$
- $V_0 = \{q_1, q_2, q_3\} = \{p_2, p_4, p_6\}.$

Spectral decimation

The key tool to prove (HSC) is the spectral decimation developed by Fukushima, Rammal, Shima and Toulouse etc.

Basic idea:
If we want to the know the eigenfunctions and eigenvalues of Δ_N *(or* Δ_D*), we just analyze its discrete form, and take limit.*

In fact, it coincides the idea which Kigami define the Laplacian on SG and general p.c.f. self-similar sets.

Discrete Laplacian Δ_m

We define discrete Laplacian Δ_m on $V_m = \bigcup_{|w|=m} F_w(V_0)$.

- Γ_m: the graph on the vertex set V_m with edge relation $\sim_m$:
  - $x \sim_m y \Leftrightarrow x \neq y$ and $\exists w$ with $|w| = m$, s.t. $x, y \in F_w(V_0)$.
- Define

$$\Delta_m u(x) = \frac{1}{\#\{y : y \sim_m x\}} \sum_{y \sim_m x} (u(y) - u(x)), \quad x \in V_m \setminus V_0.$$

- We call u_m a discrete N-eigenfunction and λ_m a discrete N-eigenvalue on V_m if

$$\begin{cases} -\Delta_m u_m(x) = \lambda_m u_m(x), & x \in V_m \setminus V_0, \\ -\frac{1}{2} \sum_{y \sim_m q_i} (u(y) - u(q_i)) = \lambda_m u_m(q_i), & q_i \in V_0. \end{cases}$$

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Spectral decimation on pre-hexagasket (Lau-Li-R)

Suppose that $\lambda_m \notin \{\frac{1}{2}, \frac{3}{2}, \frac{3\pm\sqrt{5}}{4}, \frac{3\pm\sqrt{2}}{4}\}$ and $\lambda_{m-1} = \Phi(\lambda_m)$, where $\Phi(\lambda) = \frac{2\lambda(\lambda-1)(16\lambda^2-24\lambda+7)}{2\lambda-1}$.

- If u is a discrete N-eigenfunction of Δ_{m-1} with eigenvalue λ_{m-1}, then $\exists$ an extension $\widetilde{u}$ on V_m such that $\widetilde{u}$ is a discrete N-eigenfunction of Δ_m with eigenvalue λ_m. The expression $\widetilde{u}$ on a typical V_m cell is given by

$$y_{01} = \alpha(\lambda_m)a + \beta(\lambda_m)b + \gamma(\lambda_m)c,$$
$$z_{01} = 2\big(\beta(\lambda_m) - \gamma(\lambda_m)\big)(a+b) + \delta(\lambda_m)(a+b+c),$$

where

$$\alpha(\lambda) = \eta(\lambda)^{-1}(-16\lambda^3 + 36\lambda^2 - 23\lambda + 4),$$
$$\gamma(\lambda) = \eta(\lambda)^{-1}(-\lambda + 1),$$
$$\beta(\lambda) = \eta(\lambda)^{-1}(4\lambda^2 - 7\lambda + 2), \delta(\lambda) = \eta(\lambda)^{-1},$$
$$\eta(\lambda) = (4\lambda^2 - 6\lambda + 1)(16\lambda^2 - 24\lambda + 7).$$

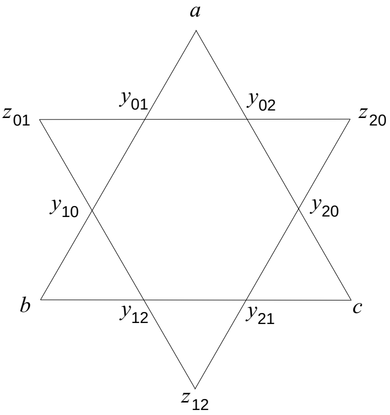

Figure: u on one cell of V_{m-1}

- Let u: discrete N-eigenfunction of Δ_m with eigenvalue λ_m. Then $u|_{V_{m-1}}$: discrete N-eigenfunction of Δ_{m-1} with eigenvalue λ_{m-1}.
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Definition of Laplacian

- $\forall f \in C(HG)$, we say $u \in \mathrm{dom}\,\Delta$ with $\Delta u = f$ on $HG \setminus V_0$ if
$$6 \cdot 14^m \Delta_m u(x) \rightrightarrows f \text{ on } V_* \setminus V_0 \text{ as } m \to \infty,$$
where $V_* = \bigcup_{m \geq 0} V_m$.
- The normal derivative at $q_i \in V_0$ of a function u on HG:
$$\partial_n u(q_i) = \lim_{m \to \infty} \left(\frac{7}{3}\right)^m \sum_{y \sim_m q_i} (u(q_i) - u(y)).$$
- $u \in \mathrm{dom}\,\Delta$ is called an eigenfunction of Neumann Laplacian with eigenvalue λ if
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According to a theorem by Shima, we can exhaust all N-eigenvalues and corresponding eigen-subspaces of Δ as:

Start from a discrete N-eigenfunction u of Δ_{m_0} with eigenvalue λ_{m_0} for a nonnegative integer m_0, and then extend u to V_* by successively using spectral decimation on pre-HG, where $\lambda_m = \Phi(\lambda_{m+1})$ for all $m \geq m_0$ with $\lambda_{m+1} = \min\{x \geq 0 : \Phi(x) = \lambda_m\}$ for all but finitely many times.

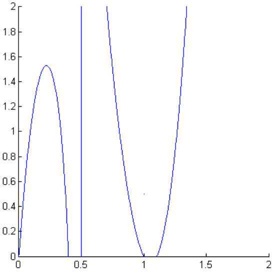

Figure: The graph of the function Φ

Theorem (Lau-Li-R)

Let $\lambda_1 = \frac{1}{4}$ *and* $\lambda_{m+1} = \min\{x > 0 : \Phi(x) = \lambda_m\}$ *for all* $m \geq 1$*. Then*

$$\lambda = \lim_{m\to\infty} 6 \cdot 14^m \lambda_m \tag{1}$$

exists, and is the second-smallest Neumann eigenvalue of Δ*. Furthermore, the multiplicity of* λ *equals 2.*

The way to disprove (HSC) on hexagasket

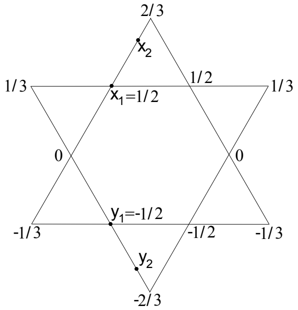

Figure: u_1 on V_1

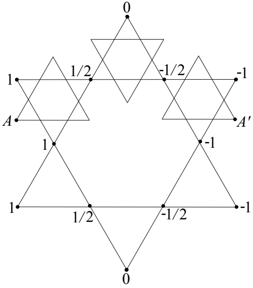

Figure: $u_1 + 2u_2$ on V_1

- u_1 attains the maximum & minimum on $\widetilde{V}_0 = \{p_1, \ldots, p_6\}$.
- $u_1 + 2u_2$ does not attains its maximum and minimum on $\widetilde{V}_0$.
- $A = 1.025, A' = -1.025$.

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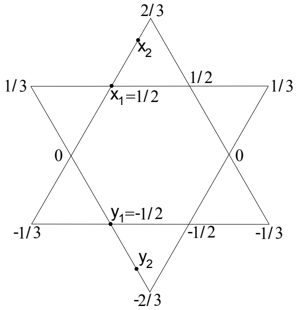

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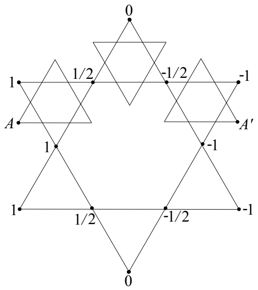

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Further remarks

The following problems are what we are doing or wish to do (with K.-S. Lau, X.-H. Li, and H. Qiu):

- Does (HSC) holds on hexagasket if we choose another IFS?

  $$F_k(x) = \frac{1}{3}(x - p_k) + p_k, \quad k = 1, \ldots, 6.$$

  In this case, $V_0 = \{p_1, \ldots, p_6\}$.

- How can we do if there is no spectral decimation?

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Thank you for attention!