Research Statement

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1. Introduction

Commutative algebra is the study of commutative rings and their modules. Many questions in commutative algebra stem from algebraic geometry, where one considers rings of functions on algebraic varieties. Such rings are noetherian, which means that all of their ideals are finitely generated. Statements about noetherian rings often reduce to the case when the ring has a unique maximal ideal; such a ring is called a local ring. This local-global perspective has made the study of local rings a fertile ground for research for at least 75 years.

The theory of modules over local rings shares many similarities with the theory of modules over graded rings with a unique homogeneous maximal ideal. Such examples of graded rings abound, and occur as important players when studying projective varieties, rings of invariants, combinatorics and commutative algebra. Many new questions concerning graded modules arise over graded rings, since the grading provides finer algebraic information.

One way to study a (graded) module is to determine how far it is from being free. Successive approximation by free modules leads to a chain complex representing the original module called a free resolution. Various questions arise when studying free resolutions: Must free resolutions necessarily be finite in length? How large are its pieces? Is there any additional structure on a resolution that will help answer the previous questions? The 'smallest' such free resolution is unique up to isomorphism, and is called the minimal free resolution of the module. The size of the free modules in the minimal free resolution are given by certain cohomology modules defined over the ring. There are very few classes of modules whose minimal free resolution is known in closed form.

My work in this area concerns primarily the study of *infinite* free resolutions. In [42], I was able to compute the minimal free resolution of modules over a fiber product of local rings (and more generally the fiber product of augmented algebras over a field); see section 2.1 for details on my work in this area. Work of Eisenbud [22] has shown that over local hypersurface rings, minimal free resolutions exhibit eventually periodic behavior. In work with Piepmeyer, Spiroff and Walker [43] we exploit this behavior and provide a deep connection between the θ function of Hochster [31] defined in terms of a free resolution of a module over a hypersurface, and the geometry of a hypersurface in projective space; see section 2.2 for details.

The chain endomorphisms of a free resolution modulo homotopy also carries the structure of a graded algebra, called the Ext algebra. The case of a free resolution of the residue field of the local ring is of particular importance; the dimensions of each graded piece give the size of a minimal free resolution of the residue field, and properties of the algebra translate into properties of the singularity of the local ring. This algebra structure is often highly noncommutative, carries subtle information about the ring, and is often difficult to compute. Using the structure of the minimal free resolution found in [42], I show that Ext algebra of a fiber product $S \times_k T$ of local rings is expressible in terms of the Ext algebra of S and T; this work is outlined in section 3.1.

I am also interested in the interaction of local algebra and rational homotopy theory. This interaction runs deep; one can find a detailed exposition of the 'looking glass' in the article by Avramov and Halperin [9]. The correspondence is not functorial, but often provides a transfer of intuition between the theories. One example of its impact on my work is seen in joint work with Avramov. Consider the connected sum $M \# N$ of manifolds M and N of the same dimension along a submanifold of one dimension less (for example, if T is a torus, then $T \# T$ is a genus two surface). The cohomology algebra of $M \# N$ is expressible via the cohomology algebras of M and N via a simple, purely algebraic construction. We extend this algebraic construction to general local rings (called the connected sum), and study the Ext algebra of a connected sum under certain desirable cohomological conditions. The details of my work in this area is highlighted in sections 3.3 and 3.2.

During the final two years as an undergraduate, I worked in software development. This experience has fostered a great interest and respect for efficient algorithms as well as computer experimentation in mathematics; many of my theorems were inpired by countless examples run on computer algebra packages such as Macaulay2 [27]. My work in this area is outlined in section 4.

2. Infinite free resolutions

In this section, $(S, \mathfrak{p}, k)$ will denote a local (or graded) ring S with unique (homogeneous) maximal ideal $\mathfrak{p}$, and $k = S/\mathfrak{p}$. Those local rings that correspond to smooth points on an algebraic variety are called regular. Pioneering work by Auslander, Buchsbaum and Serre showed that every finitely generated S-module M has a finite free resolution if and only if S is regular. Such behavior can be detected by the vanishing for large n of the cohomology modules $\operatorname{Ext}^n_S(M, k)$, which are defined in terms of a free resolution of M. In fact, these Ext modules are k-vector spaces; the dimension of $\operatorname{Ext}^n_S(M, k)$ is called the n^{th} Betti number of M over S, and is denoted $\beta^S_n(M)$. They give the 'size' of a 'smallest' free resolution of M over S, called a minimal free resolution of M.

One often collects the numbers $\beta^S_n(M)$ into a more palatable form by considering their generating function $P^S_M(t) = \sum_i \dim_k \beta^S_i(M)$, the Poincaré series of M over S. As a first approximation to finding the minimal free resolution of a module M, one may first attempt to find its Poincaré series. Any structure of the Poincaré series that exists then aids one in finding an explicit minimal free resolution.

2.1. Free resolutions over a fiber product.

One common approach to this problem is to view the S-module M through a homomorphism of local rings $\varphi : R \to S$ and construct a resolution of M over S starting with a (presumably known) resolution of M over R. In [42], I work in the opposite direction.

Theorem 2.1.1. *Let $(S, \mathfrak{p})$ and $(T, \mathfrak{q})$ be local rings with $S/\mathfrak{p} = k = T/\mathfrak{q}$, let M be a finitely generated S-module, and let R be the fiber product of S and T over the surjections $S \to k \leftarrow T$. Then a minimal free resolution of M over R can be explicitly obtained from minimal free resolutions of M and k over S and a minimal free resolution of k over T. Furthermore, this construction preserves any internal gradings that the inputs may have.*

There are few results in the literature which give infinite minimal resolutions in closed form. The minimal resolutions that result from this construction will be infinite even if the data over S and T are not, provided S and T are not fields. The size of the free modules in this resolution were determined long ago, by Kostrikin and Shafarevich when S and T are artinian and $M = k$, and by Dress and Krämer [21] in general. The determination of the differentials is new and is used in the work described in section 3.

2.2. Hochster's theta function.

If R is a hypersurface – a quotient of a regular local ring Q by a single element – then there exist modules whose minimal free resolution is infinite. However, a remarkable result of Eisenbud [22] shows that the maps in the minimal free resolution are eventually periodic of period two, so that the asymptotic homological data of an R-module is distilled into a pair of maps of free modules.

Shortly after, Hochster [31] showed that if R is an isolated hypersurface singularity (i.e., the only singular prime of R is $\mathfrak{m}$), then the function $\theta(M, N) = \operatorname{length}(\operatorname{Tor}^R_{2i}(M, N)) - \operatorname{length}(\operatorname{Tor}^R_{2i-1}(M, N))$ is independent of $i \gg 0$ for all R-modules M and N, and is biadditive along short exact sequences. Therefore, one can view θ as a pairing on the Grothendieck group of all finitely generated R-modules. Although Eisenbud's result is not needed for this, it provides a reason for θ to be independent of i at the level of resolutions of M and N, *before* taking homology.

Hochster's motivation for the study of θ was to provide conditions under which, given a pair of modules M and N with $M \otimes_R N$ of finite length, one has an inequality $\dim M + \dim N \leq \dim R$. He shows that if R is an admissible hypersurface[1] and $\theta(M, -)$ is the zero pairing, then if $M \otimes_R N$ has finite length, the dimension inequality holds. In the same paper, Hochster also reduced the direct summand conjecture to proving that the function $\theta(R/\mathfrak{p}, -)$ is zero for a certain class of mixed characteristic local rings R and primes $\mathfrak{p}$.

It is therefore interesting to find pairs of modules M, N such that $\theta(M, N) \neq 0$. A literature search shows that all currently known examples are defined over rings of odd dimension. Dao [19] has shown that θ vanishes for all two dimensional rings, and those four dimensional rings containing a field. This led him to conjecture that if R is an even dimensional isolated hypersurface singularity containing a field, then θ is identically zero.

In joint work with Piepmeyer, Spiroff and Walker [43], we study θ in the case of a graded isolated hypersurface singularity $R = k[x_0, \ldots, x_n]/(f)$. First, we show that Dao's conjecture holds in this case; that is, the θ pairing vanishes if n is even. We also show that if n is odd and k is separably closed, then θ factors

[1] R is said to be *admissible* if $R \cong T/(f)$ for a unramified regular local ring T.

through the Chern character map taking values in étale cohomology. Moreover, when $k = \mathbb{C}$ and n is odd, then $(-1)^{\frac{n+1}{2}}\theta$ is positive semidefinite, and its kernel may be identified using the Hodge-Riemann bilinear relations. Using this, we show that the $\frac{n-1}{2}^{\text{th}}$ rational Chow group modulo homological equivalence is the obstruction to the triviality of the θ pairing.

The proofs of these results bring to bear the many tools available for studying smooth hypersurfaces in projective space, as well as Eisenbud's periodicity theorem for minimal resolutions over hypersurface rings. In the future, we plan to study generalizations of θ to complete intersections, where work of Gulliksen [28], Eisenbud [22] and Avramov-Buchweitz [6, 7] will be important. Some work in this direction has already been undertaken by Dao [20].

3. The Ext algebra of a local ring

For a local ring $(S, \mathfrak{p}, k)$, the k-vector space $\mathrm{Ext}_S(k,k) = \bigoplus_n \mathrm{Ext}^n_S(k,k)$ has the structure of a graded k-algebra under the Yoneda product, and $\mathrm{Ext}_S(M,k)$ is a graded left $\mathrm{Ext}_S(k,k)$-module. These constructions are functorial *in the ring argument*.

3.1. Fiber products.

It is natural to study the behavior of the functor $\mathrm{Ext}_{-}(k,k)$ with respect to basic constructions of rings. Good answers allow one to reduce computations of Ext algebras and modules to simpler rings. A well known example is the Künneth formula, which, for k-algebras S and T, gives an isomorphism $\mathrm{Ext}_{S\otimes_k T}(k,k) \cong \mathrm{Ext}_S(k,k) \otimes_k \mathrm{Ext}_T(k,k)$ of graded k-algebras. Thus, cohomology "preserves tensor products". In a similar vein, the next result from [42] shows that cohomology also converts pullbacks into pushouts. Indeed, the classical construction of the *free product* of graded connected k-algebras, denoted here by $\sqcup$, is a coproduct in the category of such algebras.

Theorem 3.1.1. *Let $S \to k \leftarrow T$ be surjections of local rings, and let $R = S \times_k T$. The canonical homomorphism of graded k-algebras*

$$\phi\colon \mathrm{Ext}_S(k,k) \sqcup \mathrm{Ext}_T(k,k) \to \mathrm{Ext}_R(k,k) \tag{3.1.2}$$

induced by applying $\mathrm{Ext}_{-}(k,k)$ to the pullback square defining R is an isomorphism. Furthermore, if M is a finitely generated S-module, the canonical homomorphism of graded left $\mathrm{Ext}_R(k,k)$-modules below is an isomorphism:

$$\eta\colon \mathrm{Ext}_R(k,k) \otimes_{\mathrm{Ext}_S(k,k)} \mathrm{Ext}_S(M,k) \to \mathrm{Ext}_R(M,k).$$

In the graded case, these isomorphisms were proved by Backelin and Fröberg [15] using cobar constructions. This technology is not available in the local setting, which makes the computation of the product in $\mathrm{Ext}_R(k,k)$ more difficult. The structure of the resolution constructed in Theorem 2.1.1 provides a guide for this computation.

Theorem 3.1.1 provides information not present in the equalities of Poincaré series obtained by Dress and Krämer [21]. In [42], we are able to use this information to compute the depth of $\mathrm{Ext}_R(L,k)$ over $\mathrm{Ext}_R(k,k)$ for an R-module L. The notion of depth was used in [23] to study the homotopy Lie algebras of simply connected CW complexes, and of local rings. Recently, Avramov and Veliche [14] have used depth to study the relationship between the absolute and stable cohomology algebras.

In future research, I also plan on extending the results of Theorems 3.1.1 and 2.1.1 with the residue field k replaced by a local ring R_0. When the maps from R to S and R to T have desirable homological properties, Lescot [35] relates the Poincaré series of the residue fields of R, S, T, and R_0, yet leaves an explicit minimal free resolution of k, as well as the structure of $\mathrm{Ext}_R(k,k)$ unexplored. Results in this direction have applications to computing the Ext algebra of classes of Stanley-Reisner rings, as well as the question of Koszulness of Orlik-Solomon algebras. For example, the Stanley-Reisner ring of a shellable simplicial complex occurs as an iterated fiber product (provided by the shelling) where R_0 is (at worst) a squarefree monomial hypersurface.

3.2. Golod homomorphisms.

My work with Avramov [13] as well as Christensen [16] exploits the utility of surjective local homomorphisms $\varphi : R \to S$ with nice homological properties called Golod homomorphisms. Introduced by Levin [37], they have since been studied by Avramov [4], Herzog and Steurich [30], and appear in recent work of Avramov, Iyengar and Şega [10]. For the definition, let M be an S-module. Then there is a coefficient-wise inequality of formal power series

$$P_M^S(t) \preccurlyeq \frac{P_M^R(t)}{1 - t(P_S^R(t) - 1)}.$$

If equality holds, one says that M is φ-Golod, and if k is φ-Golod, one says that φ is Golod.

Examples of Golod homomorphisms include the natural projection $R \to R/(f)$ for a nonzerodivisor $f \in \mathfrak{m}^2$ [46], and the projection $R \to R/(0 : \mathfrak{m})$ for R an artinian Gorenstein ring of embedding dimension at least two [39]. In [13], Avramov and I enlarge the class of homomorphisms known to be Golod, with applications toward the connected sum of local rings in mind; see section 3.3.

In ongoing work with Christensen [16], we show that if $\varphi\colon R \to S$ is a trivially Golod homomorphism, (this includes the *strong* Golod homomorphisms of Levin [36]), a sufficiently high syzygy of every S-module M is φ-Golod. The rational expression above then provides a recurrence relation among the Betti numbers of M over S with coefficients coming from the Betti numbers of S and M over R. Such a recurrence often forces the Betti numbers of an arbitrary S-module M to strictly increase at some uniform point not depending on the module M.

As an application, we enlarge the class of rings known not to admit nonfree totally reflexive modules. An R-module M is *totally reflexive* if the natural map $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)$ is an isomorphism, and if $\operatorname{Ext}^i_R(M,R) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(M,R),R)$. This class of modules was first introduced by Auslander and Bridger [2] in their study of G-dimension; totally reflexive modules are exactly those modules of G-dimension zero. Every free module is clearly totally reflexive, and maximal Cohen-Macaulay modules over a Gorenstein ring are also totally reflexive.

The existence of nonfree totally reflexive modules over a non-Gorenstein ring has significant implications regarding the structure of the ring, as was noticed by Yoshino [48] and Christensen-Veliche [17]. There has been recent interest in finding rings that do not admit nonfree totally reflexive modules [12, 44]. The following theorem unites many of these results.

Theorem 3.2.1. *Suppose that $\varphi\colon R \to S$ is a trivially Golod [2] homomorphism such that $\operatorname{pd}_R S > 1$. Then S does not admit nonfree totally reflexive modules.*

3.3. **Connected Sums of local rings.** Before discussing the algebraic problem, I should like to describe some intuition coming from topology. When M and N are compact connected oriented n-manifolds, their connected sum $M\#N$ is formed by cutting out an n-disc from each and identifying the boundaries of the "holes" by using an orientation-reversing homeomorphism. The cohomology algebras of M, N, and $M\#N$ over $\mathbb{Q}$ are $\mathbb{Q}$-algebras and the third one is obtained from the other two by a simple, purely algebraic construction. This contruction translates into local algebra as follows.

Suppose $S \to k \leftarrow T$ are homomorphisms of local rings, and let $S \hookleftarrow J \hookrightarrow T$ be inclusions of a k-module J into S and T as ideals. In joint work with Avramov [13], we define the connected sum of S and T along J, denoted $S\#T$, to be local ring obtained by taking the quotient of $S \times_k T$ modulo the diagonal image of J. For example, if the rings S and T are artinian Gorenstein rings with residue field k, and $s \in S$ and $-t \in T$ are generators of their respective socles, their connected sum is the Gorenstein ring $(S \times_k T)/(s - t)$. Note that if S and T are the cohomology algebras of compact connected oriented n-manifolds M and N with coefficients in a field k, and J the fundamental class of each, then $S\#T$ is the cohomology algebra of $M\#N$.

In our work, we study the cohomology of $S\#T$ when k is the common residue field of S and T, and the maps $\varphi\colon S \to S' := S/J$ and $\psi\colon T \to T' := T/J$ satisfy certain desirable cohomological conditions.

Theorem 3.3.1. *Suppose that the maps φ and ψ are Golod homomorphisms, or that $\dim_k J = 1$ and the canonical map $S\#T \to S' \times_k T'$ is a Golod homomorphism. Then one has a canonical isomorphism of Hopf algebras*

$$\operatorname{Ext}_{S'}(k,k) \sqcup_{\mathsf{T}_k(\bar{J})} \operatorname{Ext}_{T'}(k,k) \longrightarrow \operatorname{Ext}_{S\#T}(k,k),$$

[2] The notion of Golod homomorphism involves the existence of *trivial Massey operations* on a differential graded algebra associated to φ. If these can be taken to be eventually zero one says φ is *trivially Golod*.

where $\mathsf{T}_k(\bar{J})$ *is the tensor algebra on* $\bar{J} := \mathrm{Hom}_k(J, k)$ *generated in degree two. In particular, one has an equality of formal power series*

$$\frac{1}{P_k^{S\#T}(t)} = \frac{1}{P_k^S(t)} + \frac{1}{P_k^T(t)} - 1 - rt^2.$$

Let K^S be the Koszul complex on a set of minimal generators for the maximal ideal of S. Its homology $H(K^S)$ has an algebra structure induced by the exterior product on K^S and carries substantial information on the ring S; a result of Avramov and Golod [8] states that S is Gorenstein precisely when $H(K^S)$ is a Poincaré algebra. In [41] we determine the structure of the algebra $H(K^{S\#T})$ as well as $H(K^{S\times_k T})$ from those of $H(K^S)$ and $H(K^T)$.

4. Computer algebra

4.1. **Macaulay2.** Macaulay2 [27] is a open source software system devoted to supporting research in algebraic geometry and commutative algebra. It includes a high level interpreter that provides the user access to a number of standard computations such as minimal free resolutions of modules, operations on ideals, and computation of Gröbner bases.

I have developed several packages for use in the Macaulay2 computer algebra system. For my undergraduate thesis I implemented the algorithm in [22] to compute Eisenbud operators on a minimal free resolution over a complete intersection, as well as an algorithm to compute modules with prescribed cohomological support [11]. Part of this work appears in joint work with myself and Jorgensen [33].

I also developed *ChainComplexExtras*, a package that extends the functionality of the *ChainComplex* object in Macaulay2; it is included with the default distribution of Macaulay2. Furthermore, together with Mike Stillman, Jen Biermann and Ben Lundell, we are currently rewriting the primary decomposition algorithms for Macaulay2. Our eventual goal is to implement the GTZ algorithm [25] that computes the primary decomposition of an ideal in $R[x]$ where R is any PID. I have been invited to participate in a Macaulay2 development workshop in January 2010 at MSRI. While there, I intend to work on the extending the functionality mentioned in the previous projects, as well as a package to compute the Koszul homology algebra of a graded ring R.

4.2. **Other interests.** The personal computer is currently undergoing a drastic change. During the past 25 years, chip makers increased a computer's processing power by increasing the clock speed of the processor so that the computer can process instructions at a faster rate. As they have reached physical barriers regarding clock speed, they have begun to increase the number of processing units (cores); many modern CPUs have up to four cores, and some of the graphics processors have upwards of three thousand cores on a single die. The user, however, only sees an improvement if the program is *multithreaded*, i.e. if the software utilizes multiple cores at once.

Although there have been some multithreaded computer algebra packages implemented [40], none of the current general open source packages have this functionality, since developing software for many cores is difficult and time consuming. However, new technology in software development such as Google's Go programming language, the Haskell multicore library, and the open source multicore specification OpenCL, are making it easier for developers to utilize the many cores that exist in today's computers. As a long term goal, I hope to use my experience in software design to help implement a multithreaded computer algebra package, or to help provide this functionality for an existing open source package.

5. Future research projects

5.1. **Betti numbers and Poincare series.** The study of the (infinite) sequence $\{\beta_n^S(M)\}$ of Betti numbers of an S-module M, and its generating function the Poincaré series $P_M^S(t)$, was initiated by the question of Kaplansky and Serre of whether $P_M^S(t)$ always represents a rational function. Eventually answered in the negative by Anick [1], the techniques developed by those searching for a solution have provided many useful tools to study other questions regarding the Betti sequence. Classes of rings for which $P_M^S(t)$ is known are few in number, but there are two important cases for which detailed information is known, namely Golod rings and complete intersections; the reason is that the *homotopy Lie algebra* $\pi^*(S)$ of such rings has a particularly nice form. The open questions mentioned below have positive answers in each of these cases.

One intriguing and surprisingly difficult problem was asked in [3]:

Question 5.1.1. *Is the sequence* $\{\beta_n^S(M)\}$ *eventually non-decreasing?*

Aside from some special cases [24, 26, 34, 45], few general results regarding this questions are known. To avoid problems with 'local' behavior of Betti numbers, questions regarding their asymptotics have also been raised, for example [5]:

Question 5.1.2. *Do there exist modules M whose Betti sequence is bounded by a polynomial, but whose Betti sequence does not satisfy* $\lim_n \frac{\beta_n^R(M)}{\alpha n^d} = 1$ *for some $\alpha \in \mathbb{R}$ and $d \in \mathbb{Z}$? Furthermore, do there exist modules whose Betti sequence grows faster than any polynomial, yet grows slower than any (proper) exponential function?*

Avramov shows [3] that for $M = k$, such intermediate growth cannot occur. In many other algebraic situations, however, such growth *is* possible; a nice treatment of examples in associative algebras, Lie algebras, and groups appears in a survey of Ufnarovskij [47]. The existence of such a dichotomy in commutative algebra would be surprising indeed, and the existence of examples would provide new insight into the asymptotic behavior of Betti numbers.

5.2. **Golod rings.** As mentioned in the previous section, Golod rings are an important class of local rings due to the vast amount of information known about the Poincaré series of its modules. Showing a local ring S is Golod, however, requires one to either compute $P_k^S(t)$ in its entirety, or to show the existence of a trivial Massey operation on the Koszul complex of S. The presence of a trivial Massey operation implies that the product on $H(K^S)$ is trivial, i.e. it satisfies $H_+(K^S) \cdot H_+(K_S) = 0$. It is interesting to therefore raise the

Question 5.2.1. *Suppose the product on $H(K^S)$ is trivial. Is S a Golod ring?*

Recent work of Jollenbeck [32] shows that if S is the quotient of a polynomial ring by a monomial ideal, then the question has a positive answer. Finding other classes of rings that enjoy this property would greatly enlarge our understanding and catalog of Golod rings.

5.3. **Koszul Homology.** If S is a local ring and I an ideal of S, let $H_i(S, I)$ denote the i^{th} Koszul homology on a minimal set of generators of I. In [18], the authors ask if the annihilators of the non-zero Koszul homologies of I are contained in the integral closure of I. While discussing with Striuli the applications of the arguments appearing in [18], we were compelled by experimental evidence to propose the following

Question 5.3.1. *Let I be an $\mathfrak{m}$-primary ideal of S, and let I^c denote the annihilator of $H_i(S, I)$. Does one have $I^c = (I^c)^c$?*

The $\mathfrak{m}$-primary assumption is necessary for a positive answer. It is shown in [18] that the question has a positive answer if I is integrally closed, or if S is a Cohen-Macaulay height two perfect ideal.

The paper [18] presents several other results regarding Koszul homology. One such result is that if I is an ideal of projective dimension $n < \infty$ and I_1 be the ideal generated by the entries of the presentation matrix of I, then $(\operatorname{Ann} H_1(R, I))^{p+1} \subseteq I$. This follows from unpublished work of Levin, which shows that $(I : I_1)^{p+1} \subseteq I$. We are able to extend Levin's theorem, showing that if I_2 is the ideal of 2×2 minors of the presentation matrix of I, then for every integer $p \geq \operatorname{pd}_R I$, one has an inclusion $(I : I_2)^{p+1} I_1^{p+1} \subseteq I$. We are currently exploring applications of this fact to questions regarding Koszul homology.

5.4. **Large homomorphisms.** One says a surjective homomorphism $\varphi\colon R \to S$ is *large* if the induced map $\varphi^*\colon \operatorname{Ext}_S(k, k) \to \operatorname{Ext}_R(k, k)$ is injective. Introduced by Levin [38], examples of large homomorphisms are the canonical projection $S \times_k T \to S$ [21, 42], and algebra retracts [29]. Given any surjective map $\varphi\colon R \to S$ and an S-module M, one has a canonical homomorphism of graded modules

$$\eta\colon \operatorname{Ext}_R(k, k) \otimes_{\operatorname{Ext}_S(k,k)} \operatorname{Ext}_S(M, k) \to \operatorname{Ext}_R(M, k).$$

In [42], I show that η is an isomorphism in the case that $\varphi\colon S \times_k T \to S$ is the canonical projection. Using this, one can compute the depth of $\operatorname{Ext}_R(M, k)$ over $\operatorname{Ext}_R(k, k)$, an invariant seen in recent work of [14].

If φ is a large homomorphism, then a result of Levin [38] shows that both the domain and codomain of η have the same Hilbert series. In the future, I will explore conditions on the large homomorphism φ that force η to be an isomorphism; this will lead to applications to computing the depth as before, and will also provide insight into computing a minimal free resolution of M as an R-module.

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