Generalizations of Conway's Thrackle Conjecture

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Background

Definition

A *thrackle* is a graph drawn in the plane such that any pair of distinct edges intersect precisely once, either at a common vertex or a transverse intersection point.

Conway conjectured that for any thrackle the number of edges does not exceed the number of vertices. This is simple to prove if all edges are straight line segments, see Erdős [1], but is open in general. Lovász, Pach, and Szegedy [4] proved that any thrackle on n vertices has at most $2n-3$ edges. This bound was improved to roughly $1.428n$ by Fulek and Pach [3].

We prove convex-geometric analogs of Conway's conjecture and establish bounds on the number of facets for higher-dimensional generalizations of thrackles. In the convex planar setting, we conjecture that a bound as in Conway's conjecture holds whenever the pairwise intersections admit a transversal set.

Planar Case

In stead of considering straight lines we consider general convex sets that pairwise intersect. The naive conjecture considering just vertices is wrong: *take the vertices of a regular 7-gon and the twenty-one triangles containing precisely one edge of the 7-gon.*

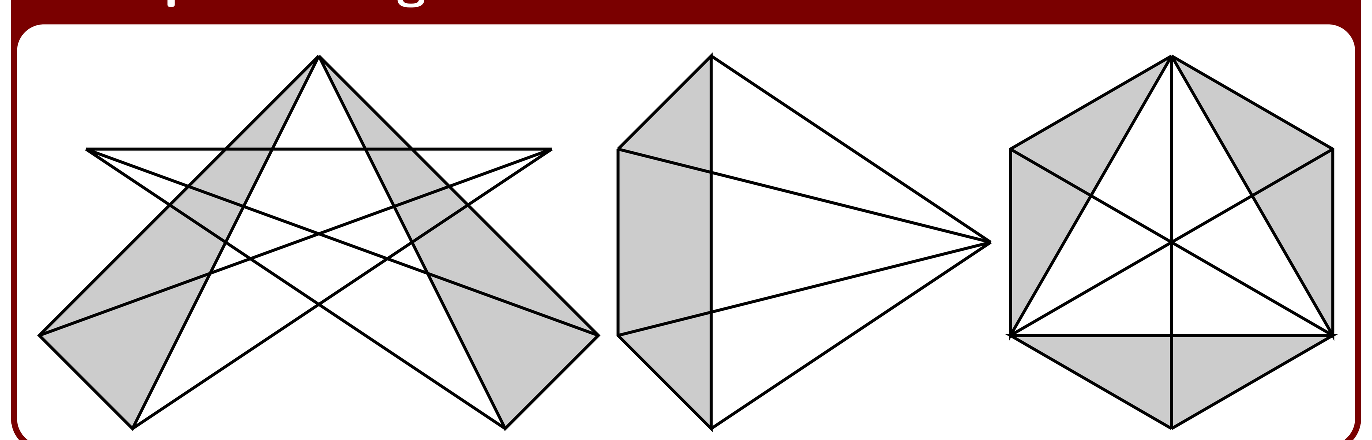
Instead, consider transversal set W, consisting of intersection points as described below. Then we can bound the number of convex sets by the total number of vertices:

Conjecture

Let $W \subseteq \mathbb{R}^2$ be a finite set of points, $V \subseteq W$ a set of n points, $C_1, \dots, C_m$ distinct convex hulls of subsets of V and $|C_i \cap C_j \cap W| = 1$ for all $i \neq j$. Then $m \leq n$.

If all the C_i have two elements, that is, they are edges, then this reduces to the linear case of Conway's thrackle conjecture.

Examples of Tight Thrackles



Planar Results

Theorem

Previous conjecture holds in the case that the vertex sets of C_i, C_j are disjoint whenever C_i, C_j are both 2-dimensional.

Proof Approach: We describe a surjection from a subset of the vertices onto the set of convex sets. Each vertex selects at most one incident set C_i using the fact that for any vertex sets can only span $(0, \pi)$ interval around the vertex if they are to intersect. We can then break down the cases based on wedge and ray placement and prove by contradiction.

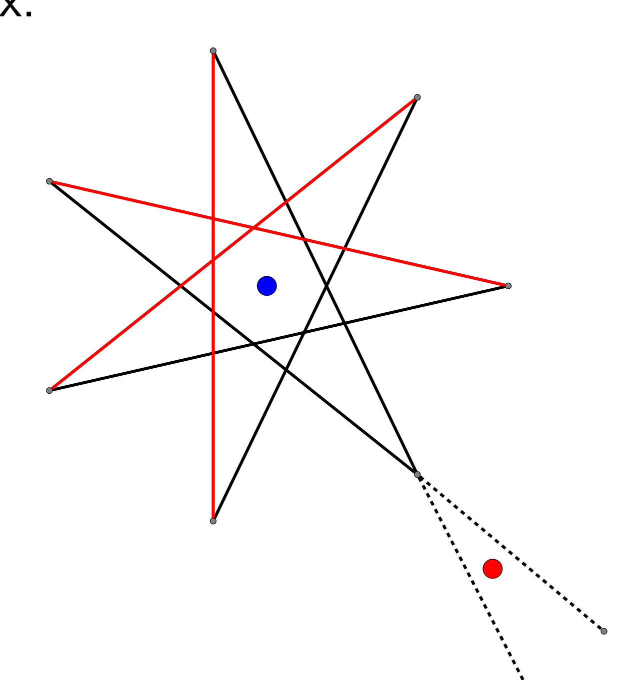
Theorem

Let $C_1, \dots, C_m$ be sets and suppose there exists a transversal of their pairwise intersections W, that is $|C_i \cap C_j \cap W| = 1$ for all $i \neq j$. Then $m \leq |W|$.

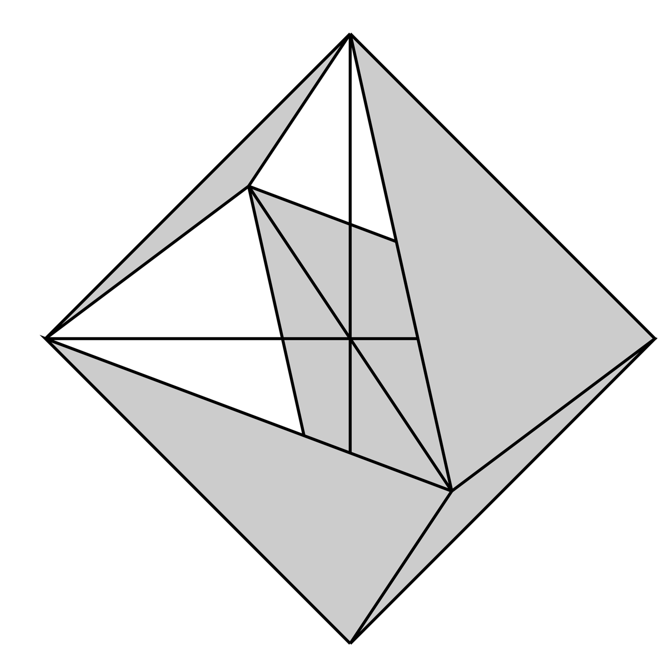
Proof Approach: If we create a graph with vertices being the sets and edges are incidence then we will see a tiling of the complete graph by complete subgraphs. The complete graph K_m cannot be decomposed into less than m complete subgraphs; see de Brujin and Erdős [2].

Examples of Higher Dimensional Thrackles

An interesting pure thrackle constructed from a traditional tight thrackle. All of the egdes are coned to the blue vertex and the indicated vertices are coned to the red vertex.



This example shows that $m \leq |W|$ does not hold for higher dimensions and is an example of a non-pure thrackle.



References

[1] Paul Erdős. "On sets of distances of n points". In: *Amer. Math. Monthly* 53.5 (1946), pp. 248–250.
[2] Paul Erdős and Nicolaas G. de Bruijn. "On a combinatorial [sic] problem". In: *Indagationes Mathematicae* 10 (1948), pp. 421–423.
[3] Radoslav Fulek and János Pach. "A computational approach to Conway's thrackle conjecture". In: *Comput. Geom.* 44 (2011), pp. 345–355.
[4] László Lovász, János Pach, and Mario Szegedy. "On Conway's thrackle conjecture". In: *Discrete Comput. Geom.* 18.4 (1997), pp. 369–376.

Higher Dimensional Thrackles

Definitions

A d-dimensional simplicial complex is *pure* if every face is contained in a d-dimensional face. A pure simplicial complex K of dimension d is called d-*thrackle* if there is a continuous map $f\colon K \longrightarrow \mathbb{R}^{d+1}$ such that

1. the restriction of f to any facet is an embedding,
2. any two facets intersect in a $(d-1)$-ball,
3. intersections between faces are *stable*, that is, there is an $\varepsilon > 0$ such that any homotopy that moves points by at most ε cannot remove the intersection.

The $(d-1)$-faces of a d-thrackle are called *ridges*. If the map f is linear on each facet then we call K *linear* d-*thrackle*.

The staright edged traditional thrackle graphs correspond to non-linear 1-thrackles.

Theorem

A linear $(d-1)$-thrackle with m facets and n ridges satisfies $dm \leq 2n$.

Proof Approach: Suppose there is a $(d-1)$-thrackle K with m facets and n ridges such that $dm > 2n$ and a a minimal counterexample. Fix an f that will embed K

- There is a ridge τ contained in at least three facets $\sigma_1, \sigma_2, \sigma_3$
- We can examine the hyperplanes that are spanned by $f(\sigma_i)$
- We can see that some hyperplane H will have an image of one of the facets on the other side of it
- This means we could remove the facet corresponding to H and get a smaller counterexample

Future Work & Acknowlegements

Higher dimensional thrackles are under more restrictions so might be an easier context under which to look at the traditional thrackle conjecture. A higher dimensional thrackle conjecture could imply the traditional thrackle conjecture.

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