Mass Equipartitions and Restricted Necklace Splittings

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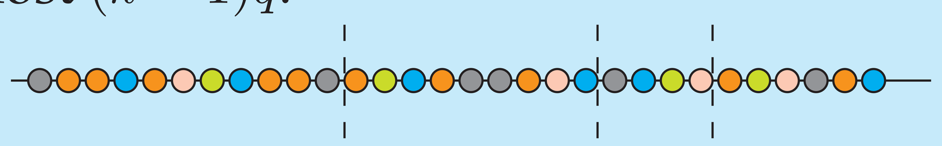
Introduction

Necklace Splittings

A group of k thieves wants to evenly divide a necklace with q types of beads among themselves such that each thief receives the same number of beads of each type. What is the smallest number of cuts that suffice in general?

Theorem (Alon) Every open necklace with q types of beads and ka_i beads of each type with $1 \leq i \leq q$ has a k-splitting of size at most $(k-1)q$.



Mass Equipartitions

Definition $\Delta(j,h)$ is the lowest dimension where any configuration of j masses can always be equiparted by h hyperplanes into 2^h orthants.

Previous Work

Ramos Conjecture

We know

$$\Delta(j,k) \geq \lceil j\frac{2^t-1}{k} \rceil$$

Ramos conjectured that this bound is tight.

Known Values of $\Delta(j,k)$

$\Delta(j,1) = j$ by the **Ham Sandwich Thm**

$\Delta(j,2) = \lceil \frac{3}{2} j \rceil$ for $j \in \{2^t-1, 2^t, 2^t+1\}$. [2, 4]

$\Delta(2,2) = 3$ and $\Delta(1,3) = 3$ [3]

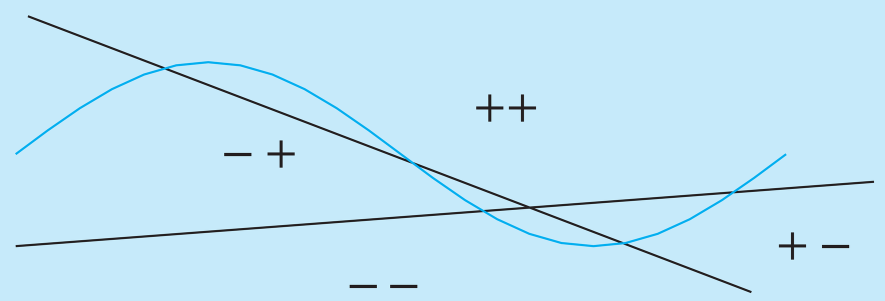
The Ramos Conjecture remains unresolved. Even $\Delta(1,4)$ is still open.

Combined Methods

Splitings for $k = 4$

Theorem For $k = 4$ thieves and all q, there exists a cyclic k-splitting of size at most $(k-1)q$.

We may view splittings of the necklace among four people as equipartitions of q masses into four orthants defined by two hyperplanes. However, the equipartition framework has the stricter property that opposite orthants (i.e. orthants that don't share a common half-space defined by any hyperplane) never receive adjacent necklace pieces.



$k = 4$, $j = 2^t$

We first prove the statement for all $j = q \in \{2^t-1,\ 2^t,\ 2^t+1\}$. Since $\Delta(2^t, 2) = 3 \cdot 2^{t-1}$, there exists a suitable equipartition in $\mathbb{R}^{3\cdot 2^{t-1}}$. We place the beads of the necklace along the moment curve γ in $\mathbb{R}^{3\cdot 2^{t-1}}$ As in [JM], place the nth bead on the point $\gamma(n)$ where $\gamma(n) = (n, n^2, \dots, n^{3\cdot 2^{t-1}})$. Define the q masses as follows:

$$A_i := \{\gamma(n)\colon \text{the nth bead is of bead type } i\}.$$

Since the moment curve is in general position, the necklace intersects each hyperplane in at most $3 \cdot 2^{t-1}$ points, yielding $2(3 \cdot 2^{t-1}) = 3 \cdot 2^t = (k-1)q$ intersection points, or cuts. Therefore, the maximum number of necklace cuts that always suffice by [AN] also suffice for a cyclic distribution of necklace pieces for $j = 2^t$ masses.

New Results

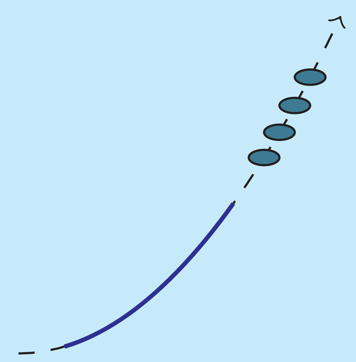
$j = 2^t - 1, 2^t + 1$.

A potential issue arises for these values of j. The bounds give minimal dimension $d = 3\cdot 2^{t-1} - 1$ and $3 \cdot 2^{t-1} + 2$ respectively, yielding up to $3 \cdot 2^t - 2$ and $3 \cdot 2^t + 4$ potential intersection points (or cuts) between the necklace and at least one of the hyperplanes.

However, a typical necklace splitting only requires at most $3(2^t-1)$ and $3(2^t+1)$ cuts respectively. Perhaps in these cases an extra cut is required to ensure a cyclic distribution of the necklace. However, this turns out not to be the case.

$k = 4$, all j

Let $j = 2^t - r$. Place the necklace along the moment curve in $\mathbb{R}^d$ where $d = 3 \cdot 2^{t-1}$. This permits at most $3 \cdot 2^t$ intersection points with two hyperplanes. Append 4 beads of r new bead types to the end of the necklace (as shown in the diagram below for $r = 1$).



This new necklace contains 2^t bead types and can be equiparted in this space. However the appended section must be parted by precisely $3r$ cuts. This leaves at most $3 \cdot 2^t - 3r = 3(2^t - r)$ cuts to equipart the original necklace. Hence, Alon's bound on the number of cuts for a general necklace splitting suffices for a cyclic splitting of any j masses as well.

Future Work

Varying k.

We may generalize the notion of cyclic necklace splittings for higher k. For $2^{t-1} < k \leq 2^t$, place each thief on a vertex of the t-dimensional hypercube. Allow two thieves to receive adjacent necklace pieces only if they share an edge of the hypercube. We call this restricted division a **binary necklace splitting**.

Conjecture Given a necklace with q types of beads and k thieves, there exists a binary necklace splitting of size $(k-1)q$.

Acknowledgements

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[1] N. Alon, *Splitting Necklaces*, Advances in Mathematics **63** (1987), 247-253.

[2] P. V. M. Blagojević, F. Frick, A. Haase, G. Ziegler *Hyperplane Mass Partitions Via Relative Equivariant Obstruction Theory*. Documenta Mathematica **21** (2016), 735-771.

[3] H. Hadwiger, *Simultane Vierteilung zweier Körper*, Arch. Math. **17** (1966), no. 3, 274-278.

[4] P. Mani-Levitska, S. Vrećica, R. T. Živaljević, *Topology and combinatorics of partitions of masses by hyperplanes*, Adv. Math. **207** (2006), 266-296.

[5] J. Matoušek, *Using the Borsuk-Ulam Theorem*, 2nd ed. Lectures on Topological Methods in Combinatorics and Geometry (Universitext, Sringer, Heidelberg, 2008).

[6] E. Ramos, *Equipartiton of mass distributions by hyperplanes*, Discrete Comput. Geom. **15** (1996), no. 2, 147-167.