Mass Equipartitions and Restricted Necklace Splittings

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Introduction

Necklace Splittings
A group of $k$ thieves wants to evenly divide a necklace with $q$ types of beads among themselves such that each thief receives the same number of beads of each type. What is the smallest number of cuts that suffice in general?

Theorem (Alon) Every open necklace with $q$ types of beads and $ka_i$ beads of each type with $1 \leq i \leq q$ has a $k$-splitting of size at most $(k-1)q$.

Mass Equipartitions
Definition $\Delta(j,h)$ is the lowest dimension where any configuration of $j$ masses can always be equiparted by $h$ hyperplanes into $2^h$ orthants.

Previous Work
Ramos Conjecture
We know $\Delta(j,h) \geq \lceil j \frac{2^h-1}{k} \rceil$
Ramos conjectured that this bound is tight.

Known Values of $\Delta(j,k)$
$\Delta(j,1) = j$ by the Ham Sandwich Thm
$\Delta(j,2) = \lceil \frac{j}{2} \rceil$ for $j \in \{2^i-1, 2^i, 2^i+1\}$.
[2, 4]
$\Delta(2,2) = 3$ and $\Delta(1,3) = 3$ [3]
The Ramos Conjecture remains unresolved. Even $\Delta(1,4)$ is still open.

Combined Methods
Splittings for $k = 4$

Theorem For $k = 4$ thieves and all $q$, there exists a cyclic $k$-splitting of size at most $(k-1)q$.

We may view splittings of the necklace among four people as equipartitions of $q$ masses into four orthants defined by two hyperplanes. However, the equipartition framework has the stricter property that opposite orthants (i.e. orthants that don’t share a common half-space defined by any hyperplane) never receive adjacent necklace pieces.

New Results

Future Work
Varying $k$. We may generalize the notion of cyclic necklace splittings for higher $k$. For $2^{i-1} \leq k \leq 2^i$, we may simplify and $\frac{k}{2}$ to a vertex of the $i$-dimensional hypercube. Allow two thieves to receive adjacent necklace pieces only if they share an edge of the hypercube. We call this restricted division a binary necklace splitting.

Conjecture Given a necklace with $q$ types of beads and $k$ thieves, there exists a binary necklace splitting of size $(k-1)q$.

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Future Work

We may generalize the notion of cyclic necklace splittings for higher $k$. For $2^{i-1} \leq k \leq 2^i$, we place each thief on a vertex of the $i$-dimensional hypercube. Allow two thieves to receive adjacent necklace pieces only if they share an edge of the hypercube. We call this restricted division a binary necklace splitting.

Conjecture Given a necklace with $q$ types of beads and $k$ thieves, there exists a binary necklace splitting of size $(k-1)q$.

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