A Minimal Irreducible Triangulation of $\mathbb{S}^3$

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While there are only finitely many triangulations (as simplicial complexes) of each surface without contractible edges [BE89], it is known that there are infinitely many such irreducible triangulations of 3-manifolds [DEG98]. Here we present a smallest irreducible triangulation $\mathcal{T}$ of $\mathbb{S}^3$ along with further interesting properties. In particular, this triangulation is positively curved but not polytopal.

Construction

Dürer's polytope is a cube where two opposite vertices are truncated and can be seen below. There is exactly one tiling of the 3-sphere by this polytope, such that the dual is a triangulation. This dual is $\mathcal{T}$.

Albrecht Dürer's Melencolia I.

An informal description of the triangulation $\mathcal{T}$ is:

1. take two disjoint Möbius strips on five vertices,
2. link them with each other
3. fill the space between them as symmetrically as possible with thirty tetrahedra.

The 1-skeleton of $\mathcal{T}$ is the same as that of the 5-dimensional cross-polytope. Thus, deleting any two non-adjacent vertices yields a subcomplex $\mathcal{T}'$ of the 4-dimensional cross-polytope, and the construction of a geometric diagram of $\mathcal{T}'$ can be seen on the right.

Removing the grey tetrahedra and red triangles from the outer tetrahedron leaves two connected components. Coning each of these with a new vertex gives the triangulation $\mathcal{T}$.

Irreducibility

In a simplicial 2-sphere one can successively contract edges to obtain the tetrahedron. It is not possible to contract an arbitrary 3-sphere to the 4-simplex. An edge is called contractible if it is not contained in any empty face, i.e. a non-face all of whose subfaces are part of the complex. The process of contracting an edge (v, w) in a simplicial complex consists of identifying the vertices v and w in every face. A simplicial complex is *irreducible* if no edge is contractible.

Theorem. Every triangulation of the 3-sphere with fewer than ten vertices, apart from the boundary of the 4-simplex, has contractible edges. The smallest instances of irreducible 3-spheres have f-vector $(10, 40, 60, 30)$ and there are six such triangulations.

This is proved by a computer enumeration using data obtained by Lutz [Lut06]. The triangulation $\mathcal{T}$ presented here is one of these six minimal irreducible triangulation of $\mathbb{S}^3$.

Positive Curvature

The dihedral angles of a regular tetrahedron are $\arccos(\frac{1}{3})$, which is slightly less than $\frac{2\pi}{5}$. Inducing the metric of a regular Euclidean tetrahedron of edge length one on every facet of a 3-dimensional triangulation introduces an angle defect (or surplus) around any edge, depending on the number of facets this edge is contained in. This number is called *valence*. The angle around an edge is less than 2π if and only if its valence is at most five. A 3-dimensional triangulation, where valences are bounded by five is called *positively curved*, since the metric above is an Alexandrov space of positive curvature.

Such a positive curvature bound from below yields a volume bound. The largest possible positively curved triangulation is the 600-cell. There are 4787 such triangulations - enumerated by Frank H. Lutz and John M. Sullivan - and 4761 of those triangulate $\mathbb{S}^3$. In addition, there are four different quotients of $\mathbb{S}^3$, which can be obtained as topological types of positively curved triangulations [LS13].

The triangulation $\mathcal{T}$ is one of these 4761 examples. Notice that a positively curved triangulation with the 1-skeleton of the 5-dimensional cross-polytope is necessarily irreducible, since each edge is contained in six 3-cycles, but at most in five triangles. Thus, every edge is contained in at least one empty triangle.

Non-Polytopality

Every triangulation of $\mathbb{S}^2$ is the boundary of a polytope. This is not true in higher dimensions.

The dual of the triangulation $\mathcal{T}$ was investigated earlier in [BS95]. Bokowski and Schuchert found four coherently oriented matroids for it, but showed these are not realizable. Thus, $\mathcal{T}$ is not the boundary complex of a 4-polytope. This does not follow from irreducibility, since it is not an obstruction to being the boundary complex of a polytope [KK87].

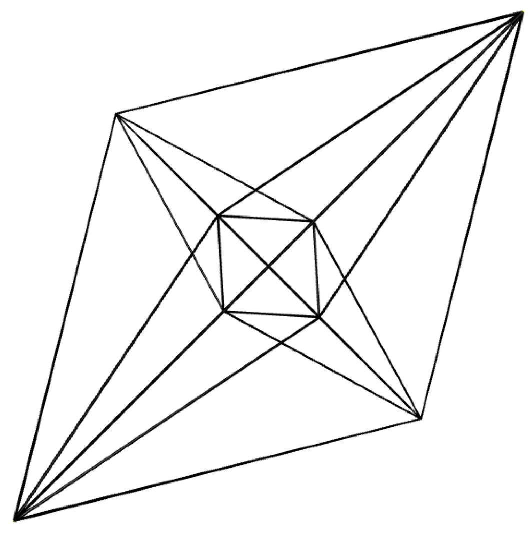
While it might be natural to assume that all positively curved triangulations are boundaries of polytopes, this is not true as $\mathcal{T}$ shows. However, $\mathcal{T}$ is embeddable into $\mathbb{R}^4$ with straight faces. Since $\mathcal{T}$ is a subcomplex of the 5-dimensional cross-polytope, it can be realized in a Schlegel diagram of the cross-polytope in $\mathbb{R}^4$.

Further Properties

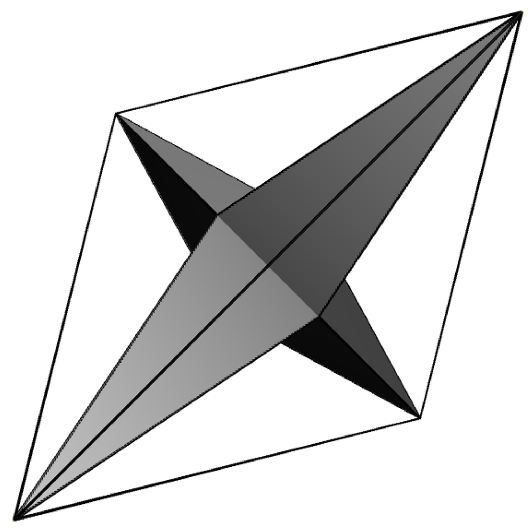
Besides being a minimal irreducible, positively curved, and non-polytopal triangulation of $\mathbb{S}^3$, $\mathcal{T}$ is

- not weakly vertex-decomposable and
- vertex-transitive.

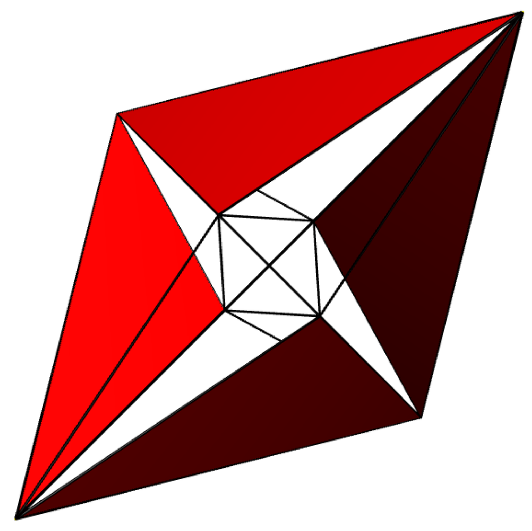
Here a pure, i.e. all facets have the same dimension, simplicial complex is *weakly vertex-decomposable* if it is a simplex or there is a vertex, such that deleting this vertex results in a weakly vertex-decomposable simplicial complex. In dimensions five and higher there are polytopes whose boundary complexes are not weakly vertex-decomposable [DLK12].



Start with (the 1-skeleton of) a Schlegel diagram of the 4-dimensional cross-polytope

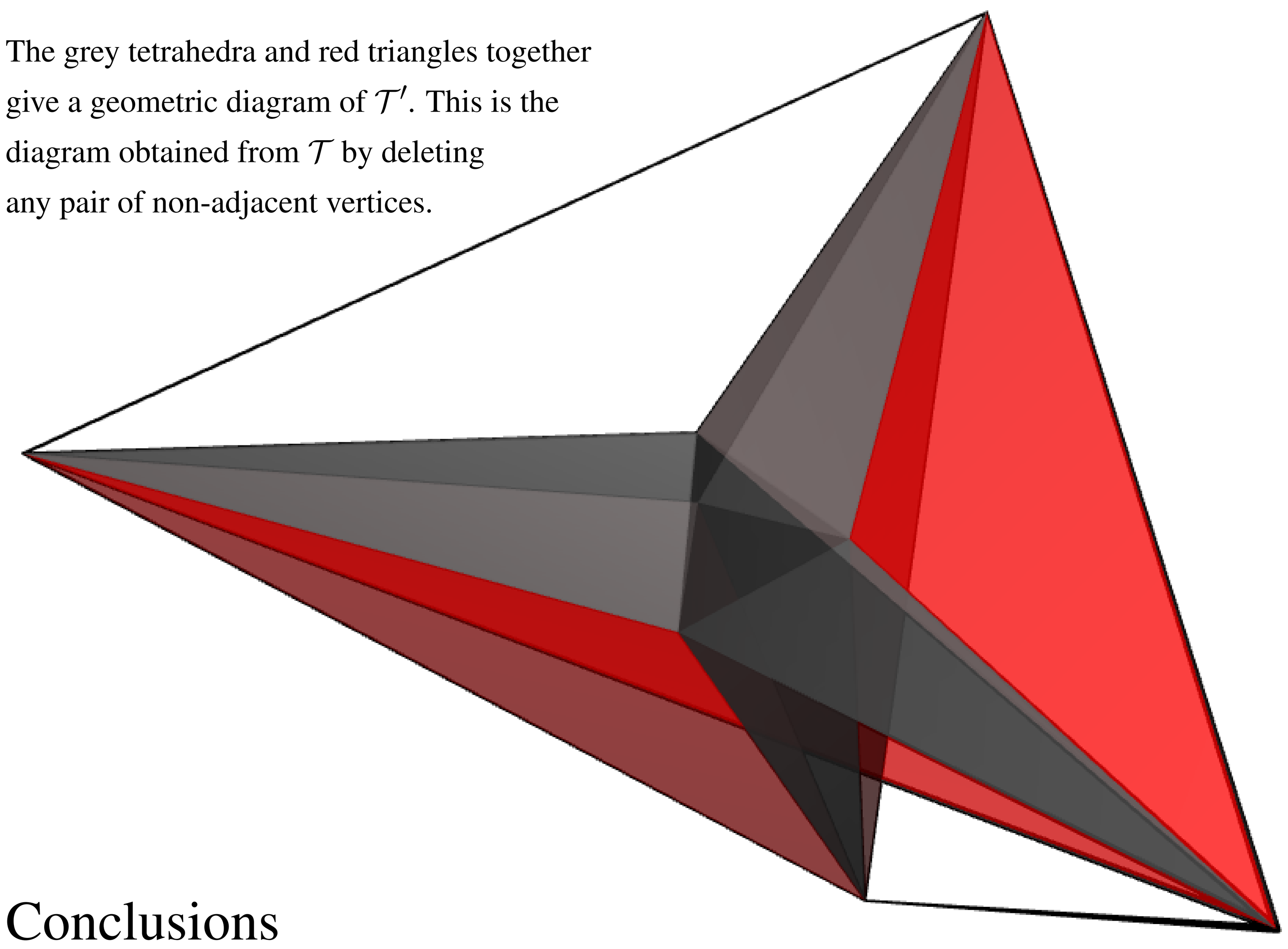


Add the middle tetrahedron, the four tetrahedra sharing a triangle with it, and the outer tetrahedron



Also add these four red triangles

The grey tetrahedra and red triangles together give a geometric diagram of $\mathcal{T}'$. This is the diagram obtained from $\mathcal{T}$ by deleting any pair of non-adjacent vertices.



Conclusions

- There are positively curved simplicial spheres which are not polytopal.
- There are simplicial 3-spheres, which are not weakly vertex-decomposable, but geometrically embeddable into $\mathbb{R}^4$.
- Triangulations of $\mathbb{S}^3$ with less than ten vertices can be edge-contracted to the boundary of the 4-simplex.

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