How to describe a moment polytope using a line bundle

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Abstract

Given a Hamiltonian group action, proving that a moment image is a convex polytope is sometimes much easier than figuring out exactly what the polytope is. But in certain cases, where the manifold is Kähler and admits a compatible holomorphic positive line bundle $\mathbb{L}$, the sections of $\mathbb{L}$ can sometimes lead to descriptions of these moment polytopes. I will discuss several results about this, proved by M. Brion, V. Guillemin and R. Sjamaar, L. O'Shea and R. Sjamaar, P. Schützdeller, and myself.

Starring:

(M, ω)	compact, connected Kähler manifold
L	holomorphic, Hermitian line bundle over M
G	compact, connected Lie group
$G_{\mathbb{C}}$	complexification of G

Assume:

$\omega = \frac{1}{2\pi i} \operatorname{curv}(\nabla)$.

$G, G_{\mathbb{C}}$ act holomorphically on (M, L) by bundle automorphisms preserving ∇.

Where's the moment map?

$\forall \xi \in \mathfrak{g}$, have two operators on $C^\infty(M, L)$: $\nabla(\xi_M), \mathcal{L}(\xi)$

Fact: (B. Kostant, 1970)
$\mathcal{L}(\xi) - \nabla(\xi_M)$ is multiplication by an $i\mathbb{R}$-valued function.

Define $\phi\colon \mathfrak{g} \to C^\infty(M)$ by

$$\phi(\xi) := \frac{1}{2\pi i}\left(\mathcal{L}(\xi) - \nabla(\xi_M)\right).$$

Can show ϕ is G-equivariant and $d\left(\phi(\xi)\right) = \omega\left(\xi_M, \cdot\right)$.

Defines a moment map!

$$\Phi\colon M \to \mathfrak{g}^*, \qquad \Phi(x)\xi = \phi(\xi)(x)$$

Also starring:

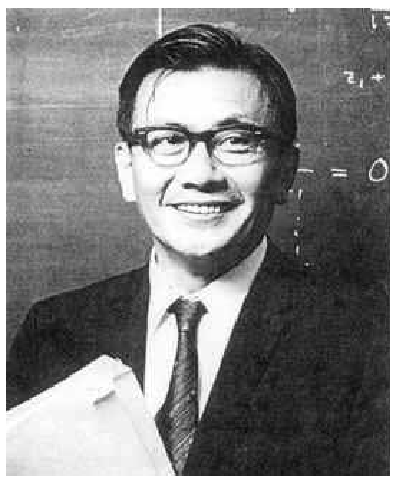
$X \subset M$	closed, irreducible, analytic subvariety
$T \subset G$	maximal torus
$\mathfrak{t}^*_+ \subset \mathfrak{t}^*$	closed positive Weyl chamber
$\Lambda \subset \mathfrak{t}^*$	weight lattice
$\Lambda_+ \subset \mathfrak{t}^*_+$	dominant weight set
$\Lambda_{\mathbb{Q}} \subset \mathfrak{t}^*$	rational points, $\Lambda_{\mathbb{Q}} := \Lambda \otimes \mathbb{Q}$
$B \subset G_{\mathbb{C}}$	Borel subgroup corresponding to $\mathfrak{t}^*_+$
$N = [B, B]$	maximal unipotent subgroup

Example:

$G = SU(n)$, $G_{\mathbb{C}} = GL(n; \mathbb{C})$,
$T = \{$diagonal elements of $G\}$,
$B = \{$upper triangular elements of $G_{\mathbb{C}}\}$,
$N = \{$elements of B with 1's on the diagonal$\}$

Kodaira Embedding Theorem: (M, L) is **ample**.

$\Rightarrow \exists$ embedding $M \hookrightarrow$ some $\mathbb{CP}^N$
(holomorphic but not usually Kähler)



So we are really working with **algebraic** things,
not just **analytic**.

The torus case

Moment map for $T \curvearrowright M$:

$$\Phi_T\colon\ M \xrightarrow{\Phi} \mathfrak{g}^* \twoheadrightarrow \mathfrak{t}^*\,.$$

Theorem: (Atiyah, 1982)
If X is preserved by $T_{\mathbb{C}}$, then $\Phi_T(X)$ is the convex hull of $\Phi_T(X^T)$.

$T \curvearrowright \Gamma(M, L)$, so

$$\Gamma(M, L) = \bigoplus_{\lambda \in \Lambda} \Gamma(M, L)_\lambda, \text{ weight space decomposition.}$$

(Note that $\dim_{\mathbb{C}} \Gamma(M, L) < \infty$.)

Alternative Statement of Atiyah's Theorem:
Let $j \colon X \hookrightarrow M$ be inclusion. If X is preserved by $T_{\mathbb{C}}$, then $\Phi_T(X)$ is the convex hull of

$$\{\lambda \in \Lambda \mid \exists s \in \Gamma(M, L)_\lambda, \ j^* s \neq 0\}.$$

The general case

Definition: The **highest weight polytope** $\mathcal{C}(X)$ of X contains $\lambda \in \Lambda_{\mathbb{Q}}$ such that:

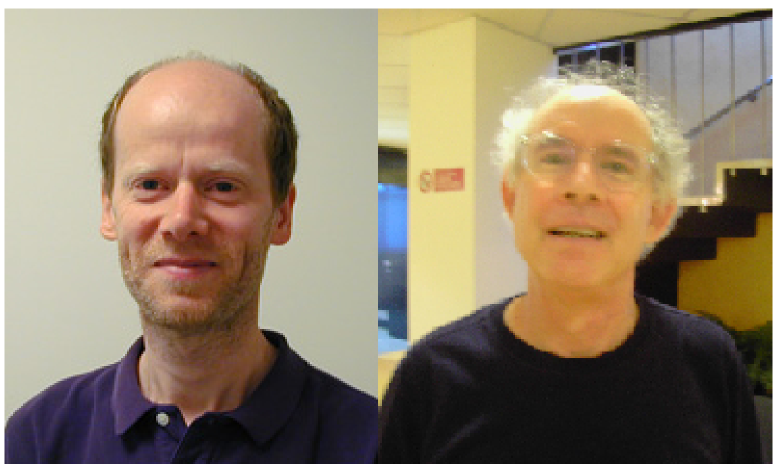
- $\exists r > 0$ with $r\lambda \in \Lambda_+$, and
- $\exists s \in \Gamma(M, L^r)_{r\lambda}$, N-invariant with $j^*s \neq 0$.

I.e. $r\lambda$ is a **highest weight** for the G-representation space $\Gamma(M, L^r)$, and $\exists$ an eigensection for $r\lambda$ not vanishing identically on X.

Put $\Delta(X) := \Phi(X) \cap \mathfrak{t}^*_+$.

Theorem: (M. Brion, 1986)
If X is $G_{\mathbb{C}}$-invariant, then $\mathcal{C}(X) = \Delta(X) \cap \Lambda_{\mathbb{Q}}$ and $\overline{\mathcal{C}(X)} = \Delta(X)$.

Theorem: (V. Guillemin and R. Sjamaar, 2006)
Even if X is only B-invariant, then Brion's result still holds.



What does $\Gamma(M, L)$ have to do with Φ?

Magic Formula:

Suppose $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ = complex Lie subalgebra,
$s \in \Gamma(M, L)$ transforms under $\mathfrak{h}$ by a character $\chi\colon \mathfrak{h} \to \mathbb{C}$.

Then $\forall \xi \in \mathfrak{h}$,

$$\boxed{\mathcal{L}(\xi_M)\|s\|^2 = (2\,\mathrm{Re}\,\chi(\xi) + 4\pi\phi(\mathrm{Im}\,\xi))\,\|s\|^2.}$$

(For (M, L^r), just replace ϕ with $r\phi$.)

Example:
Let $\mathfrak{h} = \mathfrak{g}_{\mathbb{C}}$, $\chi = 0$. Then $\forall \xi \in \mathfrak{g}$,

$$\boxed{\mathcal{L}\left((i\xi)_M\right)\|s\|^2 = 4\pi\phi(\xi)\|s\|^2.}$$

Semistability

Definition: $x \in M$ is:

- **algebraically semistable** if $\exists\, r > 0$ and $s \in \Gamma(M, L^r)^G$ with $s(x) \neq 0$.

- **analytically semistable** if $0 \in \Phi\left(\overline{G_{\mathbb{C}} \cdot x}\right)$.

Theorem:
algebraic semistability $\iff$ analytic semistability

$(\Rightarrow)$ follows from Magic Formula.
$(\Leftarrow)$ is hard. (V. Guillemin and S. Sternberg in 1982, F. Kirwan in 1984, L. Ness in 1984, R. Sjamaar in 1995.)

From this we can prove:

$$\boxed{0 \in \mathcal{C}(X) \iff 0 \in \Phi(X).}$$

The shifting trick

Given $\lambda_0 \in \Lambda_+$, want to construct (M', ω'), L', X', and $\Phi' \colon M' \to \mathfrak{g}^*$ such that

- $0 \in \Delta(X) \iff \lambda_0 \in \Delta(X')$ and
- $0 \in \mathcal{C}(X) \iff \lambda_0 \in \mathcal{C}(X')$.

Choose $\lambda_0 \in \Lambda_+$.

G_{λ_0} = stabilizer of λ_0 w.r.t. coadjoint action $G \curvearrowright \mathfrak{g}^*$

P_{λ_0} = parabolic subgroup of $G_{\mathbb{C}}$ corresponding to λ_0

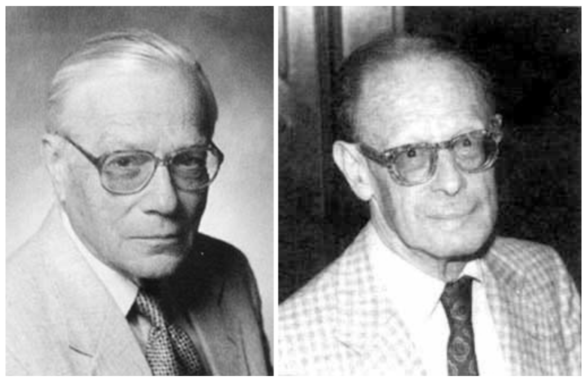
$$\begin{aligned} M_{\lambda_0} &:= G_{\mathbb{C}}/P_{\lambda_0} \text{ (**flag variety**)} \\ &\approx G/G_{\lambda_0} \text{ (**homogenous space**)} \\ &\approx G \cdot \lambda_0 \text{ (**coadjoint orbit**)} \end{aligned}$$

$$L_{-\lambda_0} := \left(G_{\mathbb{C}} \times \mathbb{C}_{-\lambda_0}\right)/P_{\lambda_0}$$

Symplectic ω' comes from $G \cdot \lambda_0$, and $\Phi_{\lambda_0} \colon M_{\lambda_0} \to \mathfrak{g}^*$ is the **negative** of the inclusion

$$M_{\lambda_0} \approx G \cdot \lambda_0 \hookrightarrow \mathfrak{g}^*.$$

Borel-Weil Theorem:
$\Gamma(M_{\lambda_0}, L_{-\lambda_0}) \cong V(\lambda_0)^*$:= dual of irreducible representation of G with highest weight λ_0.



Put

$$M' = M \times M_{\lambda_0}, \quad L' = L \boxtimes L_{-\lambda_0}, \quad X' = X \times M_{\lambda_0},$$
$$\omega' = \omega + \omega_{\lambda_0}, \quad \Phi' = \Phi + \Phi_{\lambda_0},$$

and let $G \curvearrowright M'$ diagonally.

$$\begin{aligned} \Gamma(M', L')^G &\cong (\Gamma(M, L) \otimes \Gamma(M_{\lambda_0}, L_{-\lambda_0}))^G \\ &\cong (\Gamma(M, L) \otimes V(\lambda_0)^*)^G \\ &\cong \mathrm{Hom}\,(V(\lambda_0), \Gamma(M, L))^G \\ &\cong \Gamma(M, L)^N_{\lambda_0}. \end{aligned}$$

$\Phi(x) = \lambda_0 \iff$
$\Phi'(x, \lambda) = \Phi(x) + \Phi_{\lambda_0}(\lambda_0) = \Phi(x) - \lambda_0 = 0.$

Real results

Suppose

- $(\tau, \beta) \curvearrowright (M, L)$ and $\sigma \curvearrowright G_{\mathbb{C}}$ are antiholomorphic involutions,
- β preserves ∇,
- σ preserves G, T, and B.

$$\sigma \curvearrowright G \quad \rightsquigarrow \quad \sigma \curvearrowright \mathfrak{g}, \mathfrak{g}^*$$

Compatibility with Hamiltonian action:

- **Distibution**: $\forall g \in G, x \in M$, $\tau(g \cdot x) = \sigma(g) \cdot \tau(x)$.
- **Anti-equivariance**: $\forall x \in M$, $\Phi\left(\tau(x)\right) = -\sigma\left(\Phi(x)\right)$.

σ, τ $\leftrightsquigarrow$ complex conjugation

M^{τ}, G^{σ} $\leftrightsquigarrow$ "real parts" of M, G.

Note:

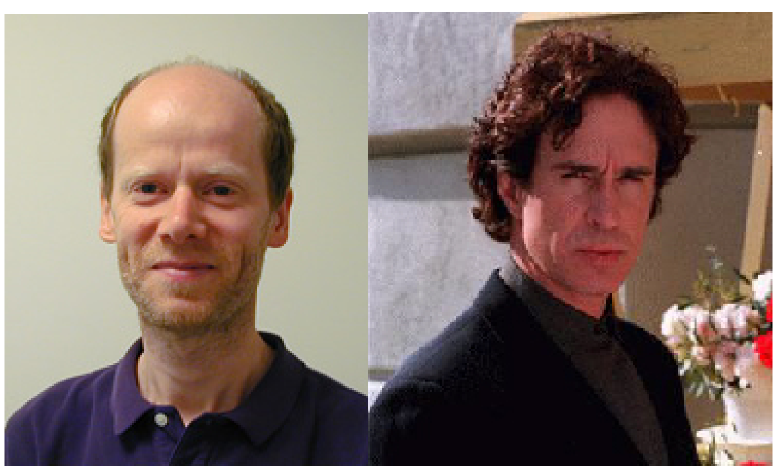
$$x \in M^{\tau} \Rightarrow \Phi(x) = \Phi\left(\tau(x)\right) = -\sigma\left(\Phi(x)\right)$$
$$\Rightarrow \sigma\left(\Phi(x)\right) = -\Phi(x)$$

So

$$\Phi\left(M^{\tau}\right) \subset \mathfrak{g}^{*}_{-1}.$$

Theorem: (L. O'Shea and R. Sjamaar, 2000)
Suppose X is preserved by $G_{\mathbb{C}}$ and τ, and X^{τ} contains a smooth point. Then

$$\Delta(X^{\tau}) = \Delta(X) \cap \mathfrak{g}^{*}_{-1}.$$



Theorem: (G., 2007) O'Shea and Sjamaar's theorem holds even if X is only preserved by B and τ.

Mantra:

The real part of the moment polytope is (ought to be) the moment polytope of the real part.

Strategy of proofs:

1. Show $\Delta(X^\tau) \cap \Lambda_\mathbb{Q} = \mathcal{C}(X^\tau) \cap \mathfrak{g}^*_{-1}$.

   (Involves variant on the shifting trick.)

2. Show $\mathcal{C}(X^\tau) \cap \mathfrak{g}^*_{-1} = \mathcal{C}(X) \cap \mathfrak{g}^*_{-1}$.

   (X^τ contains a Lagrangian submanifold,
   $\Rightarrow X^\tau$ **Zariski dense** in X.)

3. Use previous results about $\mathcal{C}(X)$ and $\Delta(X)$.

Shifting trick variant:
Need involution $\alpha \curvearrowright M_{\lambda_0} \approx G \cdot \lambda_0$.
Since this λ_0 is in $\mathfrak{g}^*_{-1}$, can define $\alpha := -\sigma \curvearrowright G \cdot \lambda_0$.

Proposition: $(G \cdot \lambda_0)^\alpha = G^\sigma \cdot \lambda_0$.

THE END

Thank you for listening.