EVERYTHING YOU'VE ALWAYS WANTED TO KNOW ABOUT CURVE-SKETCHING (BUT WERE AFRAID TO ASK)

Here we assume that $f(x)$ is a real-valued function, continuous everywhere it's defined. To sketch the graph of $f(x)$, you will generally want to follow as many of the following steps as possible.

(1) Find the domain, zeros, and intervals of positivity and negativity.

(2) Find any horizontal asymptotes.

(3) Find any vertical asymptotes.

(4) Find any critical points.

(5) Find intervals of increase and decrease.

(6) Find any local maxima and local minima.

(7) Find intervals of concavity.

(8) Find any inflection points.

(9) Put all of the information together.

Notice that the first three steps involve only $f(x)$, the next three only $f'(x)$, and the next two only $f''(x)$.

1. To find the domain, zeros, and intervals of positivity and negativity

The domain of $f(x)$ is the set of real numbers x where $f(x)$ is defined. If $f(x)$ has a denominator, the domain rejects all numbers x where the denominator is zero. If $f(x)$ has a square root, the domain rejects all numbers x where what's inside the square root is negative.

A zero of $f(x)$ is a solution to the equation $f(x) = 0$. Sometimes these are easy to find, and sometimes it is impossible. If you can find them, then you can usually find the intervals of positivity and negativity. Draw a number line with the zeros on it, labeling the number line "f". On your number line, indicate where $f(x)$ is positive and where it is negative.

2. To find horizontal asymptotes

Calculate the limits of $f(x)$ as $x \to \infty$ and as $x \to -\infty$. (These two limits are often the same, but not always.) These limits are usually calculated either by knowing what the function looks like from memory, or if you are dealing with a rational function, by dividing both the numerator and the denominator by the highest power of x appearing in the denominator.

If one or both of these limits are infinite, see if there are any **oblique asymptotes**. Recall that an oblique asymptote is a line $y = ax+b$ such that $\lim_{x\to\infty} [f(x) - (ax + b)] = 0$ or $\lim_{x\to-\infty} [f(x) - (ax + b)] = 0$. So you want to try to find a line $ax + b$ that you can subtract from $f(x)$ so that the limit of what's left as $x \to \pm\infty$ is zero.

3. To find vertical asymptotes

Check to see if there are any values of x where the denominator of $f(x)$ is 0. For each such number a, calculate the limits of $f(x)$ as $x \to a^+$ and as $x \to a^-$. If these two limits exist and are the same, then your function has a hole at this number, not an asymptote. If these two limits exist but are different, then your function has a jump at this number. If either of these two limits is infinite, then you have an actual vertical asymptote at this number. NOTE: Functions such as $\ln x$ and $\tan x$, as well as some others, have vertical asymptotes, even though they have no denominators. Be on the look-out for these.

4. To find critical points

Calculate the first derivative of $f(x)$. The critical points are the values of x where $f'(x) = 0$ or $f'(x)$ is undefined. This often amounts to setting the numerator of the function equal to 0 and solving for x, and then doing the same thing for the denominator of the function. NOTE: Functions can have any number of critical points, from none to infinitely many.

5. To find intervals of increase and decrease

Calculate the first derivative of $f(x)$. Calculate the critical points. (These are the only points where the derivative can change from positive to negative.) Draw a number line with the critical points on it, labeling the number line "f'". On your number line, indicate where $f'(x) > 0$ and where $f'(x) < 0$. The function $f(x)$ is increasing where $f'(x) > 0$, and decreasing where $f'(x) < 0$.

6. To find local minima and maxima

Recall that a local maximum is someplace where the function has a little peak, and a local minimum is someplace where the function has a little valley. A mountain might have only one highest point, but it can have lots of little peaks which are the highest points the mountain has nearby the point. There are two methods, the First Derivative Test and the Second Derivative test.

To use the **First Derivative Test**, calculate the first derivative of $f(x)$. Calculate the critical points. Draw a number line with the critical points on it. On your number line, indicate where the derivative is positive and where it is negative. The function $f(x)$ has a local maximum at critical points where the $f'(x)$ changes from positive to negative. The function $f(x)$ has a local minimum at critical points where $f'(x)$ changes from negative to positive. If the domain of $f(x)$ has endpoints, you can tell whether each is a local maximum or minimum by considering whether the function increases or decreases from the endpoint (if it is a left endpoint), or whether the function increases or decreases to it (if it is a right endpoint).

To use the **Second Derivative Test**, calculate the first derivative of $f(x)$. Calculate the critical points. Calculate the second derivative of $f(x)$. Plug the critical points into the second derivative. A critical point where $f''(x) > 0$ is a local minimum. A critical point where $f''(x) < 0$ is a local maximum. A critical point where $f''(x) = 0$ could be a local minimum, or local maximum, or neither. You have to use the First Derivative Test to find out.

7. To find intervals of concavity

Calculate the second derivative (take the derivative of $f(x)$ twice). Find the values of x where $f''(x)$ is 0 or is undefined. (These are actually the critical points of $f'(x)$.) Draw a number line with these points on it, labeling the number line "f''". On your number line, indicate where $f''(x) > 0$ and where $f''(x) < 0$. The function $f(x)$ is concave up (smiling) where $f''(x) > 0$, and concave down (frowning) where $f''(x) < 0$.

8. To find inflection points

Calculate the second derivative. Find the values of x where $f''(x)$ is 0 or is undefined. Draw a number line with these points on it, labeling the number line "f''". On your number line, indicate where the second derivative is positive and where it is negative. Inflection points are where the function changes concavity, so any point where the second derivative changes from positive to negative, or from negative to positive, is an inflection point. (Inflection points are the local minima and maxima of the first derivative.)

9. To sketch the function

After you have followed as many as the previous steps as possible, combine your number lines for $f(x)$, for $f'(x)$, and for $f''(x)$ into one final number line, labeled f. On this final number line, put all the points that are on any of the other three number lines. Label each interval on the number line positive or negative, increasing or decreasing, and concave up or concave down.

Plug some useful values of x into the function $f(x)$, obtaining several point $(x, y) = (x, f(x))$ that are contained on the graph of $f(x)$. Useful values of x include critical points and inflection points.

Sketch an xy-axes, and on the x-axis put all the points that are on your final number line. Color in the zeroes of your function, and draw dotted lines for your horizontal and vertical asymptotes. Work on one interval of your final number line at a time to sketch the curve on the axes, paying attention to zeroes and asymptotes of $f(x)$. Be sure to label your axes, any asymptotes, and the graph.

(NOTE: It may be helpful to notice that a smiling mouth has a first half which is concave up and decreasing, and a second half which is concave up and increasing. Similarly, a frowning mouth has a first half which is concave down and decreasing, and a second half which is concave down and increasing.)