

## **Condorcet's Method and Condorcet Winners**

This document provides some definitions and relations between different concepts related to the social choice procedure **Condorcet's method**. As far as I can tell, these definitions appear to be standard ones.

Recall that, given a set of alternatives, a **ballot** is an ordered list of all of the alternatives (also known as an **individual preference list**), and a **profile** is an ordered sequence of some number of ballots (also known as a **voter profile**).

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### **Definitions.**

- (a) Fix a set of alternatives and a profile. Let  $x$  and  $y$  be two alternatives. We say that  $x$  **beats**  $y$  **one-on-one** if the number of ballots on which  $x$  is ranked higher than  $y$  is larger than the number of ballots on which  $y$  is ranked higher than  $x$ . We say that  $x$  **ties**  $y$  **one-on-one** if the number of ballots on which  $x$  is ranked higher than  $y$  is equal to the number of ballots on which  $y$  is ranked higher than  $x$ .
- (b) Fix a set of alternatives. An alternative  $x$  is a **weak Condorcet winner** for a profile if, for every other candidate  $y$ , the alternative  $x$  beats or ties the alternative  $y$  one-on-one. An alternative  $x$  is a **strong Condorcet winner** (or just **Condorcet winner**) for a profile if, for every other candidate  $y$ , the alternative  $x$  beats the alternative  $y$  one-on-one.
- (c) The social choice procedure called **Condorcet's method** is defined as follows. Given the input of a set of alternatives and a profile of ballots for those alternatives, the social choice (or set of winners) under Condorcet's method is the set of weak Condorcet winners for that profile. I.e., an alternative  $x$  is a social choice under Condorcet's method if it beats or ties every other alternative one-on-one.

**Facts.** Consider a fixed set of alternatives and profile of ballots.

- (a) If there is a strong Condorcet winner, then it is unique. This is because a Condorcet winner must beat every other alternative one-on-one, which means it is never beaten. But if the Condorcet winner beats every other alternative, then every other alternative is beaten at least once, and so cannot be a Condorcet winner.

- (b) There may be multiple weak Condorcet winners, or there may be none. For example, the profile

$$\left| \begin{array}{c|c} a & b \\ b & a \\ c & c \end{array} \right|$$

has both alternatives a and b as weak Condorcet winners, while the profile

$$\left| \begin{array}{c|c|c} a & c & b \\ b & a & c \\ c & b & a \end{array} \right|$$

known as **Condorcet's voting paradox** has no weak Condorcet winners.

- (c) Any strong Condorcet winner is also a weak Condorcet winner, because if an alternative beats every other alternative one-on-one, then it is certainly true that it beats *or ties* every other alternative one-on-one.
- (d) Not every weak Condorcet winner is a strong Condorcet winner. For example, consider the following profile.

$$\left| \begin{array}{c|c|c|c|c|c} a & a & b & b & c & c \\ c & c & a & a & a & b \\ b & b & c & c & b & a \end{array} \right|$$

Here, a ties b, and a beats c, and c beats b. Therefore a is a weak Condorcet winner, but not a strong Condorcet winner, because it is tied by b.

- (e) If an alternative is a strong Condorcet winner, then Condorcet's method gives that alternative as its unique social choice; however, if an alternative is the unique social choice under Condorcet's method, it may not be a strong Condorcet winner, (although by definition it must be a weak Condorcet winner). For instance, in the profile in part (d), the alternative a is the unique social choice under Condorcet's method, but it is not a strong Condorcet winner.

To summarize, for a given set of alternatives and profile of ballots, an alternative x can be any (or none) of the following.

1. A weak Condorcet winner.
2. A strong Condorcet winner.
3. A social choice under Condorcet's method.
4. The unique social choice under Condorcet's method.

If there is a strong Condorcet winner, then there is only one. There can be multiple weak Condorcet winners. The set of social choices under Condorcet's method are exactly the weak Condorcet winners. A strong Condorcet winner is a weak Condorcet winner and hence a social choice under Condorcet's method, and is in fact the unique social choice under Condorcet's method. There are weak Condorcet winners that are not strong Condorcet winners. There are unique social choices under Condorcet's method that are not strong Condorcet winners.

**Definition.** A social choice procedure satisfies the **Condorcet Winner Criterion** (the **CWC**) if it satisfies the following.

For any set of alternatives and profile of ballots, if an alternative  $x$  is a strong Condorcet winner for that profile, then  $x$  is a social choice under the given procedure, and it is the unique social choice.

(Note that the CWC doesn't require anything special to be true if the inputted profile has no strong Condorcet winner.)