Homework 1 — solutions

**Question 1.** Create game trees for the NIM games \([1,2]\) and \([2,2]\). Assuming the players act rationally, prune the trees and determine which player will win each game.

**Solution to Question 1.**

**Game \([1,2]\):** The game tree is as follows.

```
          1 2
A's choice
  1
 B's choice
  1
```

Remember that the bottom two nodes represent wins for Player B, because Player A will have to move last. Similarly, the node on the left but one level up is a win for Player A.

First we label the three terminal nodes with the winners in those cases. Then we prune the tree, starting at the level of Player A’s last (and only) choice. If Player A chooses \(1\) or \(2\), then Player B must choose \(1\), in which case Player B wins. Therefore we carry the results B from the terminal nodes to the next highest nodes. At this point, we see that Player A has a choice between three options: \(1\), \(1,1\), and \(2\). The first choice results in a win for Player A, while the second and third result in losses. Therefore Player A would not choose the second or third options, so we cut off these branches, and carry the result A up to the root node. Hence, assuming rational play, Player A wins this game.

**Game \([2,2]\):** The game tree is as follows.

```
          2 2
A's choice
  1 2
   B's choice
  1 1
```

A's choice

```
Note that the lowest two terminal nodes are wins for Player A, and the two terminal nodes at the second-lowest level are wins for Player B.

We will begin pruning the tree by finding the semi-terminal nodes. One of these is 2 at the level of Player A’s first choice. From this node, Player B has only one option, which gives result B. Therefore we carry the result B up one level. The other semi-terminal nodes are Player B’s choices of 1 1 and 2. For each of these, Player A only has one option, both of which give result A, so we carry this result up one level in both cases.

Now we consider the node 1 2 at the level of Player A’s first choice. At this point, Player B has three options, only one of which result in a win for him. Therefore we can cut off the other two options, and carry the result B up to 1 2.

Finally, we arrive at the root of the tree. At this point, Player A has two options, but both lead to the result B! Therefore, we label the root with B, and conclude that Player B wins the game.

Remark.

1. Observe that the game tree for 1 2 is contained within the tree for 2 2. Notice also that our analysis of the 1 2 subtree within the 2 2 tree matches our analysis of the original 1 2 tree, except that the A and B labels are reversed. This is because, in the embedding of the subtree in the tree, things have shifted so that it appears as if Player B gets to choose first in the subtree.

2. In Player A’s first choice in the 2 2 game, there are two options with equal results. In the Dollar Auction, we usually used the Conservative Convention (or sometimes the Punishing Convention) to decide what a player does when confronted with a choice between equally beneficial options; however, we have not specified any such convention for the game of NIM. This did not matter for our analysis, but if we wanted to try to prove something like O’Neill’s Theorem for NIM, we would probably need to set a convention for breaking ties.