Homework 5 — solutions

ASSIGNMENT: exercises 5, 9, 12, 13, 15, and 17 in Chapter 3, (pp. 105–107), plus additional problem.

Additional Problem. Of the 18 people in our classroom, how many ways are there to choose 13 or more people? (You do not need to simplify your answer.)

Solution to Additional Problem. We can choose 13 people, or 14 people, or 15 people, or 16 people, or 17 people, or 18 people. Therefore there are

\[
\binom{18}{13} + \binom{18}{14} + \binom{18}{15} + \binom{18}{16} + \binom{18}{17} + \binom{18}{18}
\]

ways to choose 13 or more people.
(In case you are interested, this is equal to 12,616.)

Solution to Exercise 5. To construct a three-letter word, we can have 26 choices for the first letter, 26 choices for the second letter, and 26 choices for the third letter. Hence there are

\[26 \times 26 \times 26 = 17,576\]

ways to construct a three-letter word. Most of these are not even English words! Therefore, there are fewer than 17,576, which is less than 20,000, three-letter words in the English language.

Solution to Exercise 9. Since the quota is 12 votes and Luxembourg only has 1 vote, it is pivotal in a list of the six nations in the original European Economic Community if and only if the nations before Luxembourg in the list have exactly 11 votes. But Luxembourg is the only one of these nations with an odd number of votes, so a coalition of nations has an odd number of votes in total only if Luxembourg is contained in it! Since each nation can only appear once in each list, there are no lists where the nations listed before Luxembourg have exactly 11 votes. Therefore, Luxembourg is not pivotal in any list. This means that the fraction we use to compute the Shapley-Shubik Index of Luxembourg has 0 on top, and hence the fraction itself is 0.

Solution to Exercise 12. After the expansion, the weights are assigned as follows.

<table>
<thead>
<tr>
<th>Country</th>
<th>Weight</th>
<th>Country</th>
<th>Weight</th>
<th>Country</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>10</td>
<td>Belgium</td>
<td>5</td>
<td>England</td>
<td>100</td>
</tr>
<tr>
<td>Germany</td>
<td>10</td>
<td>The Netherlands</td>
<td>5</td>
<td>Denmark</td>
<td>3</td>
</tr>
<tr>
<td>Italy</td>
<td>10</td>
<td>Luxembourg</td>
<td>2</td>
<td>Ireland</td>
<td>3</td>
</tr>
</tbody>
</table>
All we need to do is show that there is at least one list where Luxembourg is a pivotal voter. We do not need to calculate the SSI, since the questions only asks us to show that the SSI is not equal to zero. This will happen when the top of the fraction that defines the SSI is not zero.

Because the quota is 41 and Luxembourg has 2 votes, so long as the nations listed before Luxembourg have a total of 39 or 40 votes, Luxembourg will be pivotal for that list.

One such list is France (10), Germany (10), Italy (10), England (10), Luxembourg (2), Belgium (5), the Netherlands (5), Denmark (3), and Ireland (3).

Therefore Luxembourg’s SSI is bigger than zero.

Solution to Exercise 13. Even if Luxembourg receives only 1 vote in the expansion, so long as the quota is still 41, Luxembourg will still be pivotal in the list we gave in the previous solution.

Solution to Exercise 15.

(a) Without Massachusetts, there is one state with 4 votes, two states with 3 votes each, one state with 2 votes, and one state with 1 vote. Because Massachusetts has 4 votes and the quota is 12, we know that Massachusetts is pivotal for a list of the states exactly when there are precisely 8 (i.e. \(12 - 4\)), 9, 10, or 11 votes before it in the list. We address each case separately.

(It might help to make a tree describing the different possible combinations of states, excluding Massachusetts, with each terminal node labeled with the number of votes that branch represents. For instance, you could have two nodes coming from the root, one labeled 0 for “no 4-vote states” and the other labeled 1 for “one 4-vote state”. Each of these could have three nodes directly below it, labeled 0, 1, and 3, and representing the number of 3-vote states.)

8 votes before MA:

- We could put \(\{\text{ME, RI, NH}\}\) first (in some order), then MA, then \(\{\text{CT, VT}\}\) (in some order). There are \(3! = 3 \times 2 \times 1 = 6\) ways to order the first three states, and \(2! = 2 \times 1 = 2\) ways to order the last two, so there are \(6 \times 2 = 12\) lists of this type.

- We could put \(\{\text{CT, ME, VT}\}\) first, then MA, then \(\{\text{RI, NH}\}\). There are also \(3! \times 2! = 6 \times 2 = 12\) lists of this type.
- We could put \{CT, RI, VT\} first, then MA, then \{ME, NH\}. There are also \(3! \times 2! = 6 \times 2 = 12\) lists of this type.

Therefore, there are \(12 + 12 + 12 = 36\) lists with exactly 8 votes before Massachusetts.

**9 votes before MA:**
- We could put \{ME, RI, NH, VT\} first, then MA, then CT. There are \(4! \times 1! = 4 \times 3 \times 2 \times 1 = 24\) lists of this type.
- We could put \{CT, ME, NH\} first, then MA, then \{RI, VT\}. There are \(3! \times 2! = 6 \times 2 = 12\) lists of this type.
- We could put \{CT, RI, NH\} first, then MA, then \{ME, VT\}. There are \(3! \times 2! = 6 \times 2 = 12\) lists of this type.

Therefore, there are \(24 + 12 + 12 = 48\) lists with exactly 9 votes before Massachusetts.

**10 votes before MA:**
- We could put \{CT, ME, RI\} first, then MA, then \{NH, VT\}. There are \(3! \times 2! = 6 \times 2 = 12\) lists of this type.
- We could put \{CT, ME, NH, VT\} first, then MA, then RI. There are \(4! \times 1 = 24\) lists of this type.
- We could put \{CT, RI, NH, VT\} first, then MA, then ME. There are \(4! \times 1 = 24\) lists of this type.

Therefore, there are \(12 + 24 + 24 = 60\) lists with exactly 10 votes before Massachusetts.

**11 votes before MA:**
The only way this can happen is if we put \{CT, ME, RI, VT\} first, then MA, then NH. There are \(4! \times 1 = 24\) lists of this type. Therefore, there are 24 lists with exactly 11 votes before Massachusetts.

Since there are six New England states, there are \(6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720\) ways to list them. Of these, we found above that Massachusetts is pivotal in \(36 + 48 + 60 + 24 = 168\) of them. Therefore

\[
\text{SSI(Massachusetts)} = \frac{168}{720} = \frac{7}{30} \approx 23.3\%.
\]
(b) Without Maine, there are two states with 4 votes each, one state with 3 votes, one state with 2 votes, and one state with 1 vote. Because Maine has 3 votes and the quota is 12, we know that Maine is pivotal for a list of the states exactly when there are precisely 9 (i.e. $12 - 3$), 10, or 11 votes before it in the list. We address each case separately.

**9 votes before ME:**

- We could put \{MA, CT, VT\} first, then ME, then \{RI, NH\}. There are $3! \times 2! = 6 \times 2 = 12$ lists of this type.
- We could put \{MA, RI, NH\} first, then ME, then \{CT, VT\}. There are also $3! \times 2! = 6 \times 2 = 12$ lists of this type.
- We could put \{CT, RI, NH\} first, then ME, then \{MA, VT\}. There are also $3! \times 2! = 6 \times 2 = 12$ lists of this type.

Therefore, there are $12 + 12 + 12 = 36$ lists with exactly 9 votes before Maine.

**10 votes before ME:**

- We could put \{MA, CT, NH\} first, then ME, then \{RI, VT\}. There are $3! \times 2! = 6 \times 2 = 12$ lists of this type.
- We could put \{MA, RI, NH, VT\} first, then ME, then CT. There are $4! \times 1 = 24$ lists of this type.
- We could put \{CT, RI, NH, VT\} first, then ME, then MA. There are $4! \times 1 = 24$ lists of this type.

Therefore, there are $12 + 24 + 24 = 48$ lists with exactly 10 votes before Maine.

**11 votes before ME:**

- We could put \{MA, CT, RI\} first, then ME, then \{NH, VT\}. There are $3! \times 2! = 6 \times 2 = 12$ lists of this type.
- We could put \{MA, CT, NH, VT\} first, then ME, then RI. There are $4! \times 1 = 24$ lists of this type.

Therefore, there are $12 + 24 = 36$ lists with exactly 11 votes before Maine.

Of the $6! = 720$ ways to list the six New England states, we found above that Maine is pivotal in $36 + 48 + 36 = 120$ of them. Therefore

$$\text{SSI(Maine)} = \frac{120}{720} = \frac{1}{6} \approx 16.7\%.$$
(c) Without New Hampshire, there are two states with 4 votes each, two states with 3 votes, and one state with 1 vote. Because New Hampshire has 2 votes and the quota is 12, we know that New Hampshire is pivotal for a list of the states exactly when there are precisely 10 (i.e. $12 - 2$) or 11 votes before it in the list. We address each case separately.

10 votes before NH:

- We could put $\{\text{MA, ME, RI}\}$ first, then NH, then $\{\text{CT, VT}\}$. There are $3! \times 2! = 6 \times 2 = 12$ lists of this type.
- We could put $\{\text{CT, ME, RI}\}$ first, then NH, then $\{\text{MA, VT}\}$. There are $3! \times 2! = 6 \times 2 = 12$ lists of this type.

Therefore, there are $12 + 12 = 24$ lists with exactly 9 votes before New Hampshire.

11 votes before NH:

- We could put $\{\text{MA, CT, ME}\}$ first, then NH, then $\{\text{RI, VT}\}$. There are $3! \times 2! = 6 \times 2 = 12$ lists of this type.
- We could put $\{\text{MA, CT, RI}\}$ first, then NH, then $\{\text{ME, VT}\}$. There are $3! \times 2! = 6 \times 2 = 12$ lists of this type.
- We could put $\{\text{MA, ME, RI, VT}\}$ first, then NH, then CT. There are $4 \times 1 = 24$ lists of this type.
- We could put $\{\text{CT, ME, RI, VT}\}$ first, then NH, then MA. There are $4 \times 1 = 24$ lists of this type.

Therefore, there are $12 + 24 + 24 = 60$ lists with exactly 9 votes before New Hampshire.

Of the $6! = 720$ ways to list the six New England states, we found above that New Hampshire is pivotal in $24 + 60 = 84$ of them. Therefore

$$SSI(\text{New Hampshire}) = \frac{84}{720} = \frac{7}{60} \approx 11.7\%.$$ 

(d) We could compute the SSI of Vermont directly, as we did those of Massachusetts, Maine, and New Hampshire. Alternatively, we could note that, because they have the same number of votes, we have $SSI(\text{Massachusetts}) = SSI(\text{Connecticut})$ and
SSI(Maine) = SSI(Rhode Island). Also, we have:

\[
SSI(MA) + SSI(CT) + SSI(ME) + SSI(RI) + SSI(NH) + SSI(VT) = 1,
\]
\[
\frac{168}{720} + \frac{168}{720} + \frac{120}{720} + \frac{120}{720} + \frac{84}{720} + SSI(VT) = 1,
\]
\[
\frac{660}{720} + SSI(VT) = 1.
\]

Hence

\[
SSI(VT) = 1 - \frac{660}{720} = \frac{720 - 660}{720} = \frac{60}{720}.
\]

Therefore

\[
SSI(Vermont) = \frac{60}{720} = \frac{1}{12} \approx 8.3\%.
\]

**Solution to Exercise 17.** First we make a table with the winning coalitions listed vertically and the voters listed horizontally, as Procedure 1 on page 86.

<table>
<thead>
<tr>
<th>Winning Coalitions</th>
<th>MA</th>
<th>CT</th>
<th>ME</th>
<th>RI</th>
<th>NH</th>
<th>VT</th>
</tr>
</thead>
<tbody>
<tr>
<td>{MA, ME, RI, NH}</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>{CT, ME, RI, NH}</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>{MA, CT, ME, VT}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>{MA, CT, RI, VT}</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>{MA, CT, ME, NH}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>{MA, CT, RI, NH}</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>{MA, ME, RI, NH, VT}</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>{CT, ME, RI, NH, VT}</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>{MA, CT, ME, RI}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>{MA, CT, ME, NH, VT}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>{MA, CT, RI, NH, VT}</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>{MA, CT, ME, RI, VT}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>{MA, CT, ME, RI, NH}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>{MA, CT, ME, RI, NH, VT}</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Sum (which equals TBP)**

<table>
<thead>
<tr>
<th></th>
<th>MA</th>
<th>CT</th>
<th>ME</th>
<th>RI</th>
<th>NH</th>
<th>VT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>10</td>
<td>8</td>
<td>8</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

Then the sum of the total Banzaf powers is \(10 + 10 + 8 + 8 + 6 + 2 = 44\), so we have the following table.
<table>
<thead>
<tr>
<th>Nation</th>
<th>TBP</th>
<th>BI</th>
<th>approximate Banzhaf percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Massachusetts</td>
<td>10</td>
<td>10/44</td>
<td>22.7%</td>
</tr>
<tr>
<td>Connecticut</td>
<td>10</td>
<td>10/44</td>
<td>22.7%</td>
</tr>
<tr>
<td>Maine</td>
<td>8</td>
<td>8/44</td>
<td>18.2%</td>
</tr>
<tr>
<td>Rhode Island</td>
<td>8</td>
<td>8/44</td>
<td>18.2%</td>
</tr>
<tr>
<td>New Hampshire</td>
<td>6</td>
<td>6/44</td>
<td>13.6%</td>
</tr>
<tr>
<td>Vermont</td>
<td>2</td>
<td>2/44</td>
<td>4.5%</td>
</tr>
</tbody>
</table>