Homework 8 — solutions

ASSIGNMENT: exercises 4, 8, and 11 Chapter 1, (pp. 36–39).

Solution to Exercise 4.

\[
\begin{array}{cccccc}
  a & b & c & d & e \\
  b & c & b & c & d \\
  e & a & e & a & c \\
  d & d & d & e & a \\
  c & e & a & b & b \\
\end{array}
\]

Condorcet’s Method: Alternative $a$ beats alternative $b$ (3 to 2) but is beaten by $c$ (4 to 1), so alternatives $a$ and $b$ cannot be winners. Alternative $c$ beats $b$ (3 to 2) but is beaten by $d$ (3 to 2), so alternative $c$ is not a winner. Alternative $d$ beats $a$ (3 to 2) but is beaten by $b$ (3 to 2), so alternative $d$ is not a winner. Finally, alternative $e$ is beaten by $a$ (3 to 2), so it is not a winner either. Therefore there are \textbf{no winners}.

Plurality Voting: Notice that each alternative occurs at the top of precisely one ballot. Therefore \textbf{all the alternatives are winners}.

The Borda Count:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Borda Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$4 + 2 + 0 + 2 + 1 = 9$</td>
</tr>
<tr>
<td>$b$</td>
<td>$3 + 4 + 3 + 0 + 0 = 10$</td>
</tr>
<tr>
<td>$c$</td>
<td>$0 + 3 + 4 + 3 + 2 = 12$</td>
</tr>
<tr>
<td>$d$</td>
<td>$1 + 1 + 4 + 3 = 10$</td>
</tr>
<tr>
<td>$e$</td>
<td>$2 + 0 + 2 + 1 + 4 = 9$</td>
</tr>
</tbody>
</table>

Therefore \textbf{c is the winner}.

The Hare System: Since each alternative appears at the top of one ballot, in the first stage we remove all of them from the ballots. Since “the alternative(s) deleted last is declared the winner”, we conclude that \textbf{all the alternatives are winners}.

Sequential Pairwise Voting with Agenda $acdeb$: We found above that $c$ beats $a$ (4 to 1). Then $d$ beats $c$ (3 to 2), but $e$ beats $d$ (3 to 2). Finally $b$ beats $e$ (3 to 2), so \textbf{b is the winner}.

The Fifth Voter is the Dictator: Since $e$ is the dictator’s top choice, \textbf{e is the winner}. 
Solution to Exercise 8.

(a) **False**
Consider the following voter profile.

<table>
<thead>
<tr>
<th>a</th>
<th>a</th>
<th>a</th>
<th>b</th>
<th>b</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
</tbody>
</table>

Since $a$ and $b$ each appear at the top of three ballots, while $c$ appears atop none, alternatives $a$ and $b$ are both social choices under plurality.

(b) **False**
In the same voter profile used above, note that $a$ and $b$ both have Borda score $2 + 2 + 2 + 1 + 1 + 1 = 9$, while $c$ has Borda score 0. Therefore $a$ and $b$ are both social choices under the Borda count.

(c) **False**
In the same voter profile used above, in the first step we would eliminate $c$, and in the second and final step we would remove both $a$ and $b$. Therefore $a$ and $b$ are both social choices under the Hare system.

(d) **False**
In the same voter profile used above, note that both $a$ and $b$ defeat $c$ one-on-one, but $a$ and $b$ tie. Therefore, for sequential pairwise voting with **any** fixed agenda, both $a$ and $b$ would be social choices.

(e) **True**
The dictator is a particular voter, who has a particular ballot. Since no ties are allowed in a ballot, there must be a unique alternative at the top of the dictator's ballot, and this alternative is the unique social choice in this dictatorship.

Solution to Exercise 11. Since the given social choice procedure is monotone, we are allowed to have one voter change his/her vote by moving the alternative $d$ (which is a social choice) up one step in his/her ballot without affecting the fact that $d$ is a social choice. Therefore, the voter with individual preference list $a, b, c, d$ (written horizontally for convenience, with preference decreasing from left to right) can change his list to $a, b, d, c$, and $d$ will remain a social choice. Again, since the procedure is monotone, this same voter can again move $d$ up one spot on his list, changing it from $a, b, d, c$ to $a, d, b, c$, and $d$ will remain a social choice. Applying this procedure once more, the voter changes his list from $a, d, b, c$ to $d, a, b, c$, and $d$ remains a social choice.