Bicycle Math

presented to the *Olivetti Club*

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Abstract

Some pretty interesting mathematics, especially geometry, arises naturally from thinking about bicycles and how they work. Why exactly does a bicycle with round wheels roll smoothly on flat ground, and how can we use the answer to this question to design a track on which a bicycle with square wheels can ride smoothly? If you come across bicycle tracks on the ground, how can you tell which direction it was going? And just what was the answer to Keith Devlin's question about the area between bicycle tracks, anyway? We will discuss the answers to these questions, and give lots of illustrations.
This talk should be accessible to undergraduates. Only an introductory knowledge of complex numbers and vector calculus will be required.

Outline

1. Bicycle wheels
   - Round bicycle wheels
   - Roulette curves
   - Polygonal bicycle wheels
2. Bicycle tracks
   - Which way did it go?
   - The area between tracks

1. Bicycle wheels

How does a bicycle roll smoothly on flat ground?

- Only wheel **edge** rolls on ground, carries rest of wheel with it.
- **As wheel rolls, center of wheel stays at constant height!**
- With axis at center of wheel, bicycle rides smoothly.

Roulette curves

Model the rolling wheel situation with **roulette curves**.

Definition

- $\mathbf{f}$ = fixed curve, $\mathbf{r}$ = rolling curve.
- $\mathbf{r}$ rolls along $\mathbf{f}$ without sliding, carrying whole plane with it, (rolling transformations).
- **Roulette curve through point** p = curve traced out by p under rolling transformations.

Roulettes generalize other curves, like cycloids and involutes.

Parametrizing roulette curves

Recall:

$$\begin{aligned}
\mathbb{C} &= \text{the plane} \\
z \in \mathbb{C} &= \text{point or vector in plane} \\
z \mapsto w \cdot z &= \text{linear transformation of plane} \\
(\text{fixed } w \in \mathbb{C}) & \quad\ (\text{rotation if } |w| = 1)
\end{aligned}$$

Parametrizing roulette curves

Parametrize **fixed** and **rolling curves** by

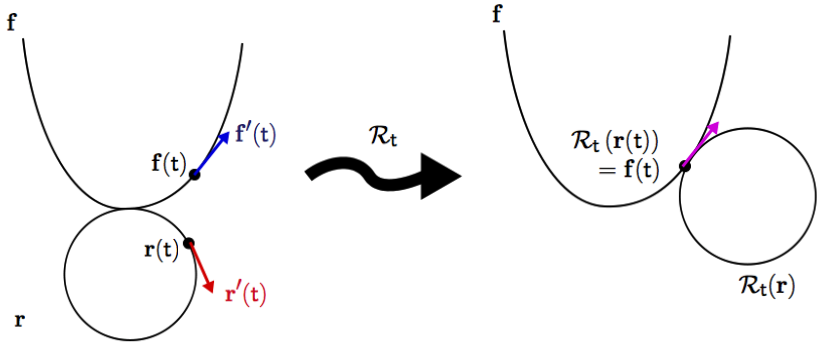
$$\mathbf{f}, \mathbf{r} \colon \mathbb{R} \to \mathbb{C}.$$

Assume:

- Curves are initially tangent:
$$\mathbf{r}(0) = \mathbf{f}(0) \text{ and } \mathbf{r}'(0) = \mathbf{f}'(0).$$
- Curves are parametrized at same speed:
$$|\mathbf{r}'(t)| = |\mathbf{f}'(t)| \neq 0 \text{ for all } t \in \mathbb{R}.$$

Parametrizing roulette curves

$\mathcal{R}_t\colon \mathbb{C} \to \mathbb{C}$, the time t rolling transformation



Parametrizing roulette curves

Definition

The **rolling transformations** generated by $\mathbf{r}$ rolling along $\mathbf{f}$ are the family

$$\{\mathcal{R}_t\colon \mathbb{C} \to \mathbb{C} \mid t \in \mathbb{R}\}$$

of rigid motions of the plane such that:

- $\mathcal{R}_t$ matches up $\mathbf{r}$ and $\mathbf{f}$ at time t:
$$\mathcal{R}_t\big(\mathbf{r}(t)\big) = \mathbf{f}(t).$$
- $\mathcal{R}_t$ maps $\mathbf{r}$ so that it is tangent to $\mathbf{f}$ at time t:
$$\tfrac{d}{ds}\,\mathcal{R}_t\big(\mathbf{r}(s)\big)\Big|_{s=t} = \mathbf{f}'(t),$$
or equivalently
$$(\mathbf{D}\mathcal{R}_t)_{\mathbf{r}(t)}\,\mathbf{r}'(t) = \mathbf{f}'(t).$$

Parametrizing roulette curves

Each $\mathcal{R}_t$ is a **rigid motion**, (preserves distance), so can be written as a rotation and translation:

$$p \mapsto \mathcal{R}_t(p) = a \cdot p + b$$

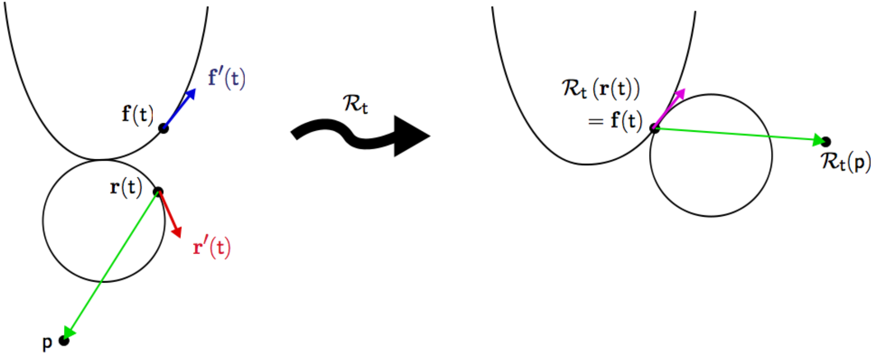
for some $a, b \in \mathbb{C}$ with $|a| = 1$.

Can use above properties of $\mathcal{R}_t$ to show that

$$\mathcal{R}_t(p) = \mathbf{f}(t) + (p - \mathbf{r}(t)) \cdot \frac{\mathbf{f}'(t)}{\mathbf{r}'(t)}.$$

Alternatively ...

Parametrizing roulette curves



- $\mathcal{R}_t$ rotates $\mathbf{r}'(t)$ to $\mathbf{f}'(t)$.
- $\mathcal{R}_t$ also rotates $p - \mathbf{r}(t)$ to $\mathcal{R}_t(p) - \mathbf{f}(t)$.

Parametrizing roulette curves

Note:

- Multiplication by $\dfrac{\mathbf{f}'(t)}{\mathbf{r}'(t)}$ is a rotation, (since $|\mathbf{f}'(t)| = |\mathbf{r}'(t)|$).
- $\dfrac{\mathbf{f}'(t)}{\mathbf{r}'(t)} \cdot \mathbf{r}'(t) = \mathbf{f}'(t)$.
- $\dfrac{\mathbf{f}'(t)}{\mathbf{r}'(t)} \cdot (p - \mathbf{r}(t)) = \mathcal{R}_t(p) - \mathbf{f}(t)$.

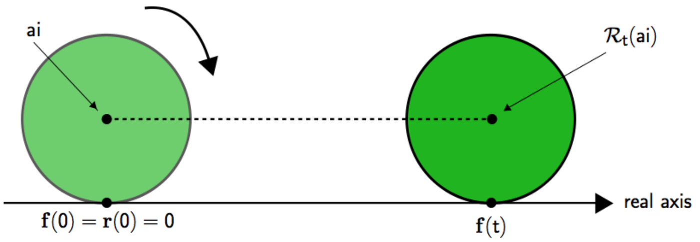
$$\boxed{\mathcal{R}_t(p) = \mathbf{f}(t) + (p - \mathbf{r}(t)) \cdot \frac{\mathbf{f}'(t)}{\mathbf{r}'(t)}.}$$

Why round wheels ride smoothly on flat ground

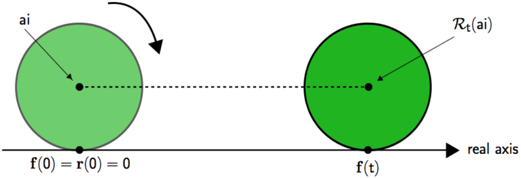
Let $\mathbf{f}$ parametrize real axis in $\mathbb{C}$ and $\mathbf{r}$ parametrize circle with radius $a > 0$ and center ai.

Steady Axle Property

The roulette through a circle's center as it rolls along a line is a parallel line, and the roulette keeps pace with the contact point between the circle and the ground line.



Why round wheels ride smoothly on flat ground



$\mathcal{R}_t(ai)$ is determined:

- vertically by ai,
- horizontally by $\mathbf{f}(t)$.

Steady Axle Equation

$$\mathcal{R}_t(ai) = ai + \operatorname{Re}\left(\mathbf{f}(t)\right).$$

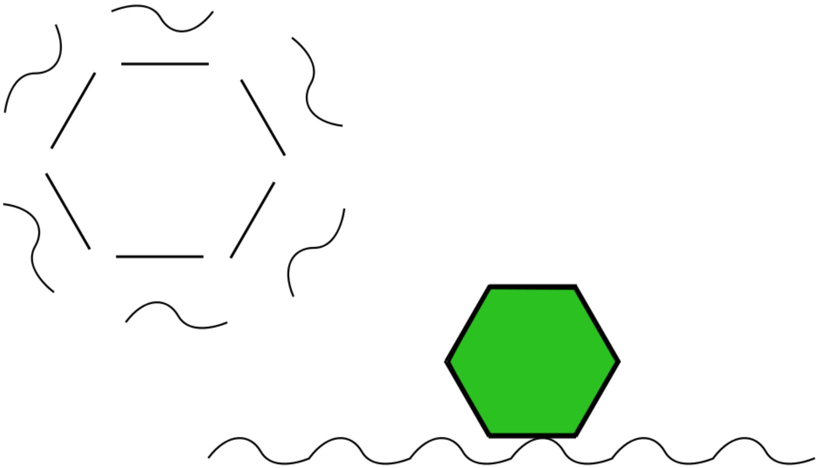
Building a track for polygonal wheels

A polygon is made up of edges glued together at vertices.

Scheme for building the track

- Build a piece of track for each polygon edge.
- Glue the pieces together.
- Check (and hope) it works.

Building a track for polygonal wheels



A piece of track for the polygon's edge

Imagine polygonal wheel lying on real axis with axle $a > 0$ units above ground.

Let $\mathbf{r}(t)$ = bottom edge of polygon $= t$,
$\mathbf{f}(t)$ = track we are trying to find.

To keep axle steady, must satisfy Steady Axle Equation:

$$\begin{aligned} ai + \mathrm{Re}\left(\mathbf{f}(t)\right) &= \mathcal{R}_t(ai) \\ &= \mathbf{f}(t) + \left(ai - \mathbf{r}(t)\right) \cdot \frac{\mathbf{f}'(t)}{\mathbf{r}'(t)} \\ &= \mathbf{f}(t) + (ai - t) \cdot \mathbf{f}'(t). \end{aligned}$$

A piece of track for the polygon's edge

Write $\mathbf{f}(t) = \alpha(t) + \beta(t)\, i$.
Then

$$ai + \mathrm{Re}\left(\mathbf{f}(t)\right) = \mathbf{f}(t) + (ai - t) \cdot \mathbf{f}'(t)$$

$$\Longleftrightarrow$$

$$\left\{ \begin{array}{rcl} a\,\alpha'(t) - t\,\beta'(t) + \beta(t) & = & a, \\ t\,\alpha'(t) + a\,\beta'(t) & = & 0. \end{array} \right.$$

Also want $\mathbf{f}(0) = \mathbf{r}(0) = 0$,
so $\alpha(0) = \beta(0) = 0$.

(system of ordinary, nonhomogeneous, first-order linear differential equations)

A piece of track for the polygon's edge

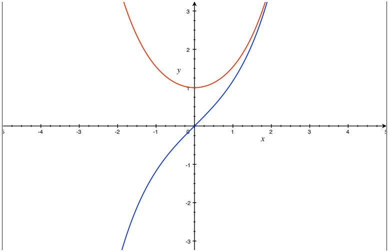
$$\left\{\begin{array}{rcl} a\,\alpha'(t) - t\,\beta'(t) + \beta(t) & = & a \\ t\,\alpha'(t) + a\,\beta'(t) & = & 0 \\ \alpha(0) = \beta(0) & = & 0 \end{array}\right.$$

Solution:

$$\alpha(t) = a\ \ln\left(t + \sqrt{a^2 + t^2}\right) - a\ \ln a = a\ \sinh^{-1}(t/a).$$

$$\beta(t) = a - \sqrt{a^2 + t^2} = a - a\cosh\left(\sinh^{-1}(t/a)\right).$$

Quick reminder



$$\color{blue}{\sinh(x) = \frac{e^x - e^{-x}}{2}} \qquad \text{and} \qquad \color{red}{\cosh(x) = \frac{e^x + e^{-x}}{2}}.$$

A piece of track for the polygon's edge

Above solution gives:

$$\begin{aligned} \mathbf{r}(t) &= t, \\ \mathbf{f}(t) &= a \sinh^{-1}(t/a) + \left[a - a\cosh\left(\sinh^{-1}(t/a)\right)\right] i. \end{aligned}$$

(Note $\mathbf{f}'(0) = \mathbf{r}'(0) = 1$ and $|\mathbf{f}'(t)| = |\mathbf{r}'(t)| = 1$.)

Reparametrize with $t = a \sinh(s/a)$. Then

$$\begin{aligned} \mathbf{r}(s) &= a \sinh(s/a), \\ \mathbf{f}(s) &= s + i \left[a - a\cosh(s/a)\right]. \end{aligned}$$

This is the graph of $y = a - a \cosh(x/a)$, an inverted **catenary curve**.

Catenaries!

Not actually a catenary.

$$y = A\,(1 - \cosh(Bx)),$$
where $A \approx 68.77$ and $B \approx 0.01$.

This is a **flattened catenary**. $(AB \neq 1)$

Catenaries!

Also not actually catenaries.

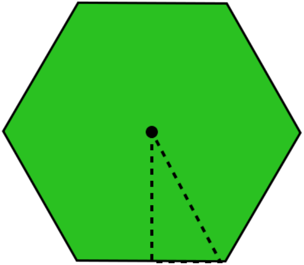
These are **canaries**.

Catenaries!

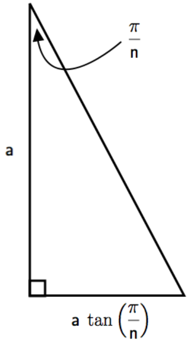
And these are **flattened canaries**.

How big is the piece of track?

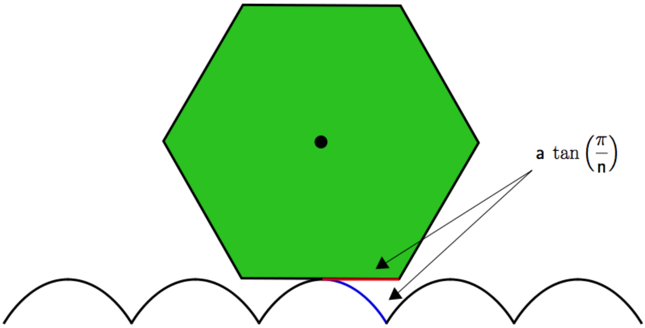
If wheel $=$ regular n-gon with axle $a > 0$ units above the ground:



Pretend this is an n-gon.



How big is the piece of track?



We reach the end of the first edge when:

$$\begin{aligned} \mathbf{r}(T) &= a\tan(\pi/n), \\ a\sinh(T/a) &= a\tan(\pi/n), \\ T &= a\sinh^{-1}(\tan(\pi/n)). \end{aligned}$$

The whole track

$T = a \sinh^{-1}(\tan(\pi/n))$.
The track is the graph of

$$y = a - a\cosh(x/a) \text{ for } -T \leq x \leq T$$

together with all horizontal translations of it by integer multiples of $2T$.

Cool fact!

As n gets larger:

- T gets smaller, each track piece gets smaller, bumps in track get smaller (although more frequent).

As polygon $\rightarrow$ circle, track $\rightarrow$ horizontal line!

Wise words

"*If the world were scallop-shaped, then wheels would be square.*"

— Krystal Allen
March 27, 2010

Things to check

- Wheel fits snuggly into gluing points of track, i.e. when wheel rolls to end of each edge, it balances perfectly on its vertex.
  TRUE, by easy computation.

- Wheel never gets stuck, i.e. wheel only intersects track *tangentially*.
  FALSE for triangular wheels!
  But **TRUE** for square wheels, pentagonal wheels, hexagonal wheels, etc.
  (Computation is hard.)

Demonstrations

2. Bicycle tracks

Key facts about bicycles

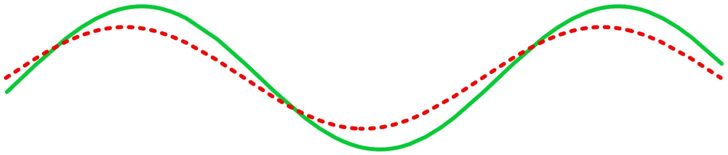
- Front and rear wheels stay fixed distance apart.
- Rear wheel **always** points towards the front wheel.

Therefore:

Key Property of Bicycle Tracks

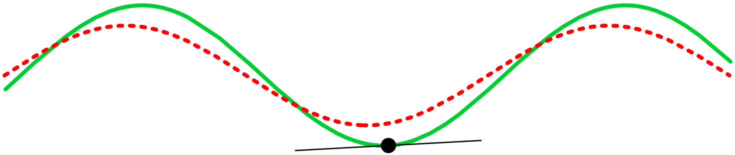
Tangent line to rear wheel track always intersects front wheel track a fixed distance away.

Which way did it go?



- Which is the rear wheel track?
- Which way did the bicycle go?

Which way did it go?

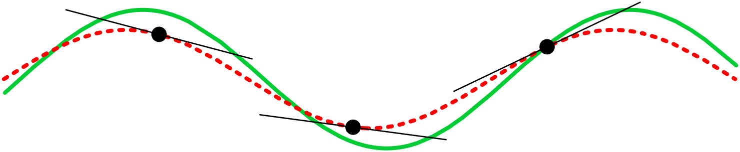


- Is the **green** (solid) one the rear wheel track?

Nope!

(Unless the bicycle is GIGANTIC!)

Which way did it go?

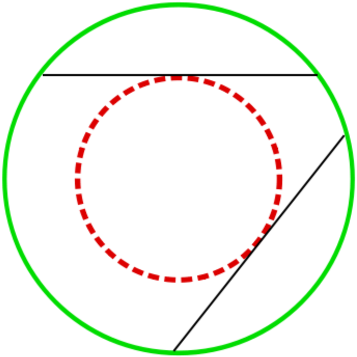


- Is the **red** (dashed) one the rear wheel track?

Yes!

And it went to the right!

Tracks where this doesn't work



Devlin's question

At the end of his talk last semester, Keith Devlin asked:

How can you find the area between front and rear bicycle tire tracks?

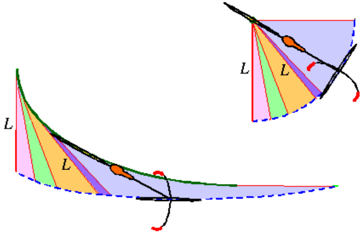
Devlin's question

Devlin's question

The answer:

- The area is swept out by the bicycle, i.e. by tangent vectors to the rear wheel track.
- Rearrange the tangent vector **sweep** into a tangent vector **cluster**!

Devlin's question



The answer:

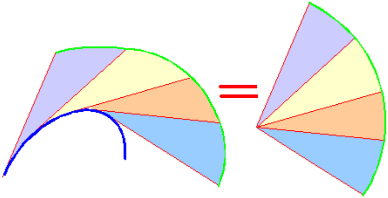
The area between front and rear bicycle tire tracks is

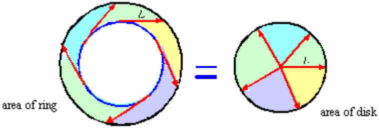
$$\frac{\theta}{2\pi} \cdot \pi L^2 = \frac{1}{2}\,\theta L^2,$$

where $L =$ distance between tires and $\theta =$ change in bicycle's angle.

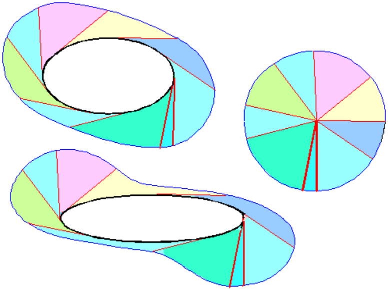
Visual Calculus

This is an example of **Visual Calculus**, developed by Mamikon Mnatsakanian. (See the Wikipedia article and notes by Tom Apostol.)





Mamikon's theorem



Mamikon's Theorem

The area of a tangent sweep is equal to the area of its tangent cluster, regardless of the shape of the original curve.

THE END

Thank you for listening.

(Don't forget to tip your waiters and waitresses.)

THE END

Special thanks to

The Amazing Andrew Cameron

for all of his help with the square-wheel track!

And HAPPY BIRTHDAY tomorrow!