The Lie Bracket and the Commutator of Flows

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Abstract

There is a famous and fundamental result that states that the flows of two vector fields commute if and only if the Lie bracket of these two vector fields is the zero vector field. A natural next step is to try to relate exactly how much the Lie bracket deviates from zero to the extent to which the flows fail to commute. Can we describe the Lie bracket entirely in terms of the failure of the flows to commute?

In this talk, I will present one answer to this question, in the form of a famous (and slightly strange) equation. To build up to this, I will give a brief and general introduction to vector fields, flows, and the Lie bracket. If time permits, I will discuss the context where this formula seems most natural, that of Lie groups.

Some basic knowledge of manifolds will be helpful, but not really necessary. At the end of the talk, I will be open to any questions about comic books.

Contents

1 Manifolds

Greetings, fellow mammals. Thank you for coming. The main character of today's story will be $\mathcal{M}$, a smooth manifold. A smooth manifold is a type of topological space that locally looks like some $\mathbb{R}^n$, in such a way that we can do calculus on it. The adjective "smooth" means that we can take as many derivatives as our hearts desire.

For those of you not familiar with manifolds, you can imagine a smooth surface in $\mathbb{R}^3$, like a sphere. If you're unfamiliar with surfaces, you can imagine an open subset of $\mathbb{R}^n$. If you're unfamiliar with that, I pity you.

2 The Tangent Bundle

2.1 Tangent Vectors and the Tangent Bundle

Each point p in a manifold comes with its own little vector space attached, called the tangent space of $\mathcal{M}$ at p, denoted $T_p\mathcal{M}$. If $\mathcal{M}$ is connected, then all the tangent spaces have the same dimension. If you imagine gathering up all of these vector spaces together, we get something called the tangent bundle, denoted and defined by

$$T\mathcal{M} := \bigsqcup_{p \in \mathcal{M}} T_p\mathcal{M}.$$

The tangent bundle is actually a smooth manifold also, with twice the dimension of the original. You can think of the tangent bundle as a bunch of vector spaces pinned together by the original manifold, although there's a little more to it than that.

2.2 Differentials of Maps

If $f\colon \mathcal{M} \to \mathcal{N}$ is a smooth map between smooth manifolds, then we have an induced map $df\colon T\mathcal{M} \to T\mathcal{N}$, and

$$df|_{T_p\mathcal{M}}\colon T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$$

is a linear map for each $p \in M$.

3 Vector Fields

3.1 As Tangent Vectors

The other major characters of our play are vector fields. A vector field is a smooth map $X\colon \mathcal{M} \to T\mathcal{M}$ such that $X(p) \in T_p\mathcal{M}$ for all $p \in \mathcal{M}$. Think of a vector field as laying down a vector in each tangent space, in such a way that the vectors vary smoothly as you change tangent spaces.

3.2 $C^\infty(\mathcal{M})$

Another way to characterize vector spaces makes use of the set $C^\infty(\mathcal{M})$ of smooth real-valued functions on $\mathcal{M}$, which has an extremely strong supporting role here. (For me, these functions never seem to be the main characters, but they always provide important context for understanding and interpreting other things.) This set is a vector field over $\mathbb{R}$, infinite-dimensional, and it is an algebra over $\mathbb{R}$. You can add them, and multiply them, and multiply them by scalars.

3.3 As Derivations

A derivation on $C^\infty(\mathcal{M})$ is a linear map $D\colon C^\infty(M) \to C^\infty(M)$ that obeys the Leibniz rule, or product rule:

$$D(f \cdot g) = f \cdot D(g) + D(f) \cdot g.$$

A derivation is like a directional derivative with a vector in its pocket. If you give me a function and a point, I can take it's directional derivative at that point in the direction of the vector. Each vector field X induces a derivation D_X just like this. If you give me a function and a point, I can take it's directional derivative at that point in the direction given by the X at that point. It's an important, and non-trivial, fact that every derivation induces a smooth vector field.

4 Flows

4.1 A Single Pebble - Integral Curves

Like I mentioned before, a vector field gives a direction at every point in the manifold. Now, let's imagine dropping a pebble into the manifold, and letting it swim around in the direction given by the vector field. Think of a stream with a current, and a pebble dropped into the stream. (You also have to imagine that the pebble doesn't sink. Maybe a twig is better. In any case, the analogy isn't perfect.) Paying attention to time as the pebble flows, and imagining the pebble flowing backward also, we obtain a curve $c(t)$ that goes through the point at time zero, and whose derivative at every time always matches the vector field. This is called an integral curve of the vector field through the point. The integral curve may not be able to continue forever. It may only be able to travel for a limited time before the currents get too confusing. Remember that this is a very smart pebble. It doesn't just take off in one direction and keep going. At every point, it re-evaluates where it should be heading. Thus, the integral curve may only be defined on a small neighborhood of zero.

4.2 A Lot of Pebbles - The Full Flow

Now imagine that we do something drastic. We take a whole bunch of pebbles, and then drop one in over every point, all at the same time, and let the pebbles all follow the current of the vector field. For each t that is small enough that all of the pebbles have been flowing until time t, we obtain a smooth map

$$\Phi_X^t \colon \mathcal{M} \to \mathcal{M},$$

called the time t flow of the vector field X. The image of the time t flow of X gives a snapshot of the mass exodus of all the points of $\mathcal{M}$ in the direction of X at time t. Notice that Φ_X^0 is the identity.

4.3 The Semigroup Property - Flows are Diffeomorphisms

The flow has a very important property. We can ask what happens if we allow a pebble to flow for time s, and then flow a little more for time t. The result is the same as if the pebble had flowed for time $s+t$. Thus $\Phi_X^t \circ \Phi_X^s = \Phi_X^{s+t}$, so long as both sides are defined. This means that

$$\Phi_X^{-t} \circ \Phi_X^t = \Phi_X^0 = \Phi_X^t \circ \Phi_X^{-t},$$

so Φ_X^t has a smooth inverse, so it is what we call a diffeomorphism, an isomorphism in the world of smooth manifolds. This is very cool, and very useful.

5 Two Vector Fields

5.1 Do the Flows Commute?

Now we up the ante, and look at two vector fields, X and Y. These give two different directions for a pebble to follow. We can ask the following very interesting question. Suppose we drop the pebble, order it to flow with X for time t, and then with Y for time t. On the other hand, what if the pebble had flowed first with Y and then with X? Does the pebble end up in the same place both times? With maps, we are asking if

$$\Phi_Y^t \circ \Phi_X^t = \Phi_X^t \circ \Phi_Y^t.$$

Do the flows commute?

5.2 An Example of Yes - Constant Vector Fields

In general, the answer is ... no! Let's take a simple example. Suppose our manifold is $\mathbb{R}^n$, and our two vector fields are constant. Then the flows of these vector fields through a given point are straight lines in space, and our situation is represented by a parallelogram in space. Since this parallelogram is closed, the flows commute. How is this different from more general vector fields?

5.3 Why the General Answer is No: Our Pebbles Are Smart

The constant vector field case is what would happen if our pebbles were stupid, and picked a direction and just marched off that way. For varying vector fields, with our smart pebbles, directions are being re-evaluated every instant. There's no reason to think that things would bend and twist together at the end.

6 Lie Brackets Give the Answer

A good question is this. Is there an easy way to tell whether the flows of two vector fields commute? The answer is no. I'm just kidding. The answer is yes.

6.1 Derivation Definition

We take our two vector fields and build a new one. If this new vector field is constantly zero, then the flows commute. If not, then they don't. Recall that X and Y induce derivations

D_X and D_Y. We define a map $D_{[X,Y]}\colon C^\infty(M) \to C^\infty(M)$ by

$$D_{[X,Y]} = D_X \circ D_Y - D_Y \circ D_X.$$

It's pretty clear that this is a linear map, but there's no reason to assume that it satisfies the product rule. There's absolutely no reason to assume that this is a derivation, except, perhaps, that it actually is! Since derivations correspond to vector fields, this defines a new vector field $[X, Y]$, called the Lie bracket of X and Y.

6.2 Lie Derivative Definition

There is another way to take two vector fields and produce a new one, called Lie differentiation. Fortunately, we end up with the same thing. The Lie derivative of Y in the direction X is equal to the Lie bracket of X and Y,

$$\mathscr{L}_X Y = [X, Y].$$

6.3 The Basic Theorem

So, we have

$$\Phi^t_Y \circ \Phi^t_X = \Phi^t_X \circ \Phi^t_Y \qquad \text{if and only if} \qquad [X, Y] = 0.$$

(The derivation definition of the Lie bracket makes it particularly obvious why it has something to do with commutativity. This is far less obvious from the Lie derivative definition.)

Recall that the flow of X commutes with itself, so $[X, X] = 0$. This is pretty cool. It's actually also true that $[X, Y] = -[Y, X]$.

But $[X, Y]$ is a whole new vector field, and it's not always identically zero. What then? If we know how much the flows fail to commute, can we figure out what the Lie bracket is?

7 Commutators

7.1 The Commutator Map for a Group

To answer this question, we change gears a little bit. Let G be a group. Then we have the map $K\colon G \times G \to G$ given by $(g, h) \mapsto h^{-1}g^{-1}hg$, the commutator map of G. If two elements commute, then their image under this map is zero. Thus, this map takes two elements and somehow measures their failure to commute. Back to manifolds, we notice that the set $\mathrm{Diff}(\mathcal{M})$ of diffeomorphisms $\mathcal{M} \to \mathcal{M}$ form a group under composition. Therefore we can talk about the commutator of two diffeomorphisms. Remember that flows are diffeomorphisms.

7.2 The Commutator Map of the Flows

If we simplify the equation $\Phi_Y^t \circ \Phi_X^t = \Phi_X^t \circ \Phi_Y^t$, we obtain $\Phi_Y^{-t} \circ \Phi_X^{-t} \circ \Phi_Y^t \circ \Phi_X^t = \mathrm{Id}$. The left side of this equation is the commutator of the flows of X and Y at time t. Thus $[X, Y] = 0$ if and only if the commutator of the flows of X and Y is the identity. It stands to reason, then, that maybe there is some equation out there that relates these two things.

7.3 Fix a Point - The Important Curve

7.4 The First Derivative

Let's fix a point $p \in \mathcal{M}$. Then we can define a curve c_p through p by

$$c_p(t) = \Phi_Y^{-t} \circ \Phi_X^{-t} \circ \Phi_Y^t \circ \Phi_X^t(p) = K(\Phi_X^t \circ \Phi_Y^t)(p).$$

We have a curve through p, and a vector $[X, Y](p)$ at p. It would be the most natural thing in the world if the velocity of this curve were the same as the vector.

It's too good to be true. It is a very important, and very confusing, fact that $c_p'(0) = 0$. The curve c_p is equal to the constant curve at p up to first order.

7.5 The Second Derivative

What actually is true is that

$$[X, Y](p) = \frac{1}{2} c_p''(0).$$

The careful among you may notice that there's something fishy going on here. The left side is a tangent vector to M, but $c_p'(t)$ is a curve in the tangent bundle $T\mathcal{M}$, so $c_p''(0)$ is a tangent vector to $T\mathcal{M}$. The left side belongs to $T\mathcal{M}$, but the right side belongs to $T(T\mathcal{M})$! How does this equation even make sense?!?!?

8 Interpretation - Other Ways of Thinking of Tangent Vectors

8.1 Point Derivations

We have to step back for a bit. Remember that vector fields can be given by derivations. In the same way, individual vectors can be given by point derivations. A point derivation at p is a linear map $D_p \colon C^\infty(\mathcal{M}) \to \mathbb{R}$ such that

$$D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g).$$

You can think of a point derivation as a directional derivative with a point and a tangent vector from that point in its pocket. The tangent space $T_p\mathcal{M}$ can be identified with the space of all point derivations at p.

8.2 Infinitesimal Curves as Point Derivations

But it gets even better. Every tiny curve that goes through p defines a tangent vector at p by its velocity. Since tangent vectors at p are point derivations at p, each tiny curve must also define a point derivation. It's actually very easy to see how this works. Let c be a curve with $c(0) = p$. Then c acts on the function f at p by

$$f \mapsto (f \circ c)'(0).$$

Notice that this is a real number, because $f \circ c$ is just a single-variable calculus function. You can check pretty quickly that this obeys the product rule.

9 The Weird Point Derivation, Given by the Second Derivative

Now, the claim is that we can define a point derivation at p using the second derivative of c_p at 0, but this actually depends on the crucial but bizarre fact that $c_p'(0) = 0$. Define a point derivation D_p at p by $D_p(f) := (f \circ c_p)''(0)$. This is certainly a linear function $C^\infty(\mathcal{M}) \to \mathbb{R}$, but proving that it obeys the product rule depends on our bizarre fact. We calculate

$$\begin{aligned}(f \cdot g \circ c_p)''(t) &= \frac{d}{dt}(f \cdot g \circ c_p)'(t) \\ &= \frac{d}{dt}(f \circ c_p)'(t)\,(g \circ c_p)(t) + (f \circ c_p)(t)\,(g \circ c_p)'(t) \\ &= (f \circ c_p)''(t)\,(g \circ c_p)(t) + (f \circ c_p)'(t)\,(g \circ c_p)'(t) + (f \circ c_p)'(t)\,(g \circ c_p)'(t) + (f \circ c_p)(t)\,(g \circ c_p)''(t) \\ &= (f \circ c_p)''(t)\,(g \circ c_p)(t) + 2(f \circ c_p)'(t)\,(g \circ c_p)'(t) + (f \circ c_p)(t)\,(g \circ c_p)''(t).\end{aligned}$$

To make this obey the product rule, we'd like to get rid of that pesky middle term. But since $c_p'(0) = 0$, using the chain rule we have

$$(f \circ c)'(0) = df_{c(0)}\, c'(0) = df_{c(0)}\, \vec{0} = 0,$$

where the last fact follows from the fact that derivatives are linear maps. Therefore at $t = 0$, the middle term in that horrible calculation drops out, and we obtain

$$\begin{aligned}D(f \cdot g) = (f \cdot g \circ c_p)''(0) &= (f \circ c_p)''(0)\,(g \circ c_p)(0) + (f \circ c_p)(0)\,(g \circ c_p)''(0) \\ &= D(f) \cdot g(p) + f(p) \cdot D(g).\end{aligned}$$

Yay!!!

Of course, it's not at all obvious why we might expect the tangent vectors given by $c_p''(0)$ to vary smoothly as we vary the point p. The most immediate reason that I trust this to be true is that these vectors are the same as twice the vectors given by the Lie bracket, which is definitely smooth.