HAMILTONIAN ACTIONS IN GENERALIZED COMPLEX GEOMETRY

TIMOTHY E. GOLDBERG

These are notes for a talk given in the Lie Groups Seminar at Cornell University on Friday, September 25, 2009. In retrospect, perhaps a more accurate title would have been *An introduction to Dirac and generalized complex geometry*.

ABSTRACT. Generalized complex (GC) geometry is a relatively new field of study that has its roots in Dirac geometry, and can be seen as generalizing Poisson, complex, and symplectic geometry. Many concepts and methods from symplectic geometry have been generalized and applied to GC geometry. For instance, in 2006 Yi Lin and Susan Tolman developed a notion of generalized Hamiltonian actions and generalized moment maps. These maps have proven to have many properties analogous to their symplectic counterparts.

In this talk, I will give an introduction to GC geometry and generalized Hamiltonian actions, and discuss the reduction of a GC manifold by a generalized Hamiltonian action.

CONTENTS

Date: September 25, 2009.

INTRODUCTION

Dirac geometry was developed by Theodore Courant [1] and Alan Weinstein. Generalized complex (GC) geometry was developed by Nigel Hitchin [3], Marco Gualtieri [2], and Gil Cavalcanti. A GC structure on a manifold is a Dirac structure satisfying an additional condition.

1. DIRAC STRUCTURES

For any n-dimensional real vector space V, the associated vector space $\mathbb{V} := V \oplus V^*$ carries a natural non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of signature (n, n), defined by

$$\langle u + \alpha, v + \beta \rangle := \frac{1}{2}\left(\alpha(v) + \beta(u)\right)$$

for all $u + \alpha, v + \beta \in \mathbb{V}$. We will use the same notation for the complex linear extension of this bilinear form to the complexification

$$\mathbb{V}_{\mathbb{C}} := (V \oplus V^*) \otimes \mathbb{C}.$$

These bilinear forms will henceforth be referred to as the ***standard metrics*** on $\mathbb{V}$ and $\mathbb{V}_{\mathbb{C}}$.

Definition 1.1. A linear subspace E of $\mathbb{V}$ or $\mathbb{V}_{\mathbb{C}}$ is ***isotropic*** if $\langle e, e' \rangle = 0$ for all $e, e' \in E$. A maximal isotropic subspace of $\mathbb{V}$, (i.e. isotropic with real dimension n), is called a ***linear Dirac structure*** on V. A maximal isotropic complex subspace of $\mathbb{V}_{\mathbb{C}}$, (i.e. isotropic with complex dimension n), is called a ***complex linear Dirac structure*** on V.

Let $T \colon V_1 \to V_2$ be a linear map between finite-dimensional real vector spaces. Elements $w_1 = u_1 + \alpha_1 \in \mathbb{V}_1$, $w_2 = u_2 + \alpha_2 \in \mathbb{V}_2$ are ***related by*** T, denoted $w_1 \sim_T w_2$, if $T(u_1) = u_2$ and $T^*(\alpha_2) = \alpha_1$. If $E_1 \subset \mathbb{V}_1$, $E_2 \subset \mathbb{V}_2$ are real linear Dirac structures, then T is a ***Dirac map*** if

$$E_1 = \{w_1 \in \mathbb{V}_1 \mid \exists w_2 \in E_2 \text{ such that } w_1 \sim_T w_2\}.$$

The definitions are similar for the complex case.

Let M be an n-dimensional manifold. The ***Pontryagin bundle***, or ***generalized tangent bundle***, of M is

$$\mathbb{T}M := TM \oplus T^*M.$$

The standard metric on each fiber of $\mathbb{T}M$ induces a smoothly varying metric on the bundle, and by complex linear extension on the bundle's complexification

$$\mathbb{T}_{\mathbb{C}}M := (TM \oplus T^*M) \otimes_{\mathbb{R}} \mathbb{C}.$$

The Lie bracket of vector fields defines a skew-symmetric bilinear bracket on $\Gamma(TM)$, the space of smooth sections of $TM \to M$. This can be extended to a skew-symmetric bilinear bracket on $\Gamma(\mathbb{T}M)$, called the ***Courant bracket***, defined by

$$\left[X+\alpha, Y+\beta\right] := [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}\,d\big(\beta(X) - \alpha(Y)\big).$$

This extends complex linearly to a bracket on $\Gamma(\mathbb{T}_{\mathbb{C}}M)$.

Definition 1.2. A smooth vector subbundle $E \subset \mathbb{T}M$ is an ***almost Dirac structure*** if E_x is a linear Dirac structure on $\mathbb{T}_xM$ for all $x \in M$. An almost Dirac structure E is a ***Dirac structure*** if $\Gamma(E)$ is closed under the Courant bracket.

A smooth map $\phi\colon M_1 \to M_2$ between manifolds is a ***Dirac map*** with respect to almost Dirac structures $E_1 \subset \mathbb{T}M_1$ and $E_2 \subset \mathbb{T}M_2$ if

$$\phi_*\colon T_xM_1 \to T_{\phi(x)}M_2$$

is a Dirac map with respect to $E_1|_x$ and $E_2|_{\phi(x)}$, for all $x \in M_1$.

Complex analogues of these notions are defined in the obvious ways.

Example 1.3.

(1) Let $B \in \Gamma\left(\wedge^2(TM)\right)$ be a smooth bivector on M. We will view this as a smoothly-varying skew-symmetric bilinear form on covectors, which hence induces a bundle map $B^\sharp\colon T^*M \to TM$ defined over each $x \in M$ by $\alpha \mapsto B(\alpha,\cdot)$ for $\alpha \in T^*_xM$, under the identification $(T^*_xM)^* \cong T_xM$. Define $E_B \subset \mathbb{T}M$ by

$$E_B = \mathrm{graph}(B) = \{B(\alpha) + \alpha \mid \alpha \in T^*M\}.$$

This is an almost Dirac structure on M, and it is a Dirac structure if and only if $[B,B] = 0$, (the Schouten–Nijenhuis bracket), i.e. if and only if B is a ***Poisson structure*** on M.

(2) Let $\Omega \in \Gamma\left(\wedge^2(T^*M)\right)$ be a smooth differential 2-form on M. This induces a bundle map $\Omega^\flat\colon TM \to T^*M$ defined over each $x \in M$ by $v \mapsto \Omega(v,\cdot)$ for $v \in T_xM$. Define $E_\Omega \subset \mathbb{T}M$ by

$$E_\Omega = \mathrm{graph}(\Omega) = \{u + \Omega^\flat(u) \mid u \in TM\}.$$

This is an almost Dirac structure on M, and it is a Dirac structure if and only if $d\Omega = 0$, i.e. if and only if Ω is a ***presymplectic structure*** on M.

2. GENERALIZED COMPLEX STRUCTURES

Definition 2.1. Let V be a finite-dimensional real vector space. A ***linear GC structure*** on V is a complex linear Dirac structure $E \subset \mathbb{V}_{\mathbb{C}}$ such that $\mathbb{V}_{\mathbb{C}} = E \oplus \overline{E}$, (i.e. $E \cap \overline{E} = \{0\}$). Let $\rho\colon V_{\mathbb{C}} \oplus V_{\mathbb{C}}^* \to V_{\mathbb{C}}$ be the natural projection. The ***type*** of this GC structure is the complex codimension of $\rho(E)$ in $V_{\mathbb{C}}$:

$$\operatorname{type}(E) = \dim_{\mathbb{C}} V_{\mathbb{C}} - \dim_{\mathbb{C}} \rho(E).$$

Let M be a manifold. An ***almost generalized complex structure*** on M is a smooth vector subbundle $E \subset \mathbb{T}_{\mathbb{C}}M$ such that E_x is a linear GC structure on $\mathbb{T}_{\mathbb{C},x}M$ for each $x \in M$. An almost GC structure E is a ***generalized complex structure*** if $\Gamma(E)$ is closed under the Courant bracket.

Equivalently, an almost GC structure is an complex almost Dirac structure E on M such that the intersection $E \cap \overline{E}$ is the image of the zero section of $\mathbb{T}_{\mathbb{C}}M \to M$, and a GC structure is an almost GC structure such that $\Gamma(E)$ is Courant-closed.

For each $x \in M$, the ***type*** of an almost GC structure $E \subset \mathbb{T}_{\mathbb{C}}M$ at x is the type of E_x in $\mathbb{T}_{\mathbb{C},x}M$:

$$\operatorname{type}_x(E) = \operatorname{type}(E_x) = \dim_{\mathbb{C}} T_{\mathbb{C},x}M - \dim_{\mathbb{C}} \rho(E),$$

where here ρ is the natural projection $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M \to T_{\mathbb{C}}M$.

Proposition 2.2. *Let V be a real vector space. There is a natural bijective correspondence between the following two structures.*

(1) *Linear GC structures on V, $E \subset \mathbb{V}_{\mathbb{C}}$.*
(2) *Orthogonal linear $\mathcal{J}\colon \mathbb{V} \to \mathbb{V}$ such that $\mathcal{J}^2 = -\mathrm{id}$.*

Idea of proof.

(1) $\Rightarrow$ (2): Define $\mathcal{J}_{\mathbb{C}}\colon \mathbb{V}_{\mathbb{C}} \to \mathbb{V}_{\mathbb{C}}$ with $(+i)$-eigenspace E and $(-i)$-eigenspace $\overline{E}$. Since $\mathbb{V}_{\mathbb{C}} = E \oplus \overline{E}$ and $\mathbb{V} = \{x + \overline{x} \mid x \in \mathbb{V}_{\mathbb{C}}\}$, we can write

$$\mathbb{V} = \{e + \overline{e} \mid e \in E\}.$$

For all $e \in E$, we have $\mathcal{J}_{\mathbb{C}}(e + \overline{e}) = ie - i\overline{e} = ie + \overline{ie} \in \mathbb{V}$. Therefore $\mathcal{J}_{\mathbb{C}}$ preserves $\mathbb{V}$. Put $\mathcal{J} := (\mathcal{J}_{\mathbb{C}})|_{\mathbb{V}}$.

(2) ⇒ (1): Let E be the $(+i)$-eigenspace of the complex linear extension $\mathcal{J}_{\mathbb{C}}$ of $\mathcal{J}$. □

Proposition 2.3. *Let* M *be a manifold. There is a natural bijective correspondence between the following two structures.*

(1) *Almost GC structures on* M, $E \subset \mathbb{T}_{\mathbb{C}}M$.
(2) *Orthogonal bundle maps* $\mathcal{J}\colon \mathbb{T}M \to \mathbb{T}M$ *such that* $\mathcal{J}^2 = -\mathrm{id}$.

Example 2.4.

(1) Let ω be a non-degenerate differential 2-form on M, and let $\omega^\flat\colon TM \to T^*M$ be the associated bundle map. Because ω is non-degenerate $\omega^\flat$ is an isomorphism. We denote its inverse by $\omega^\sharp$. Define the map $\mathcal{J}_\omega\colon \mathbb{T}M \to \mathbb{T}M$ by
$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^\sharp \\ \omega^\flat & 0 \end{pmatrix}.$$
The $(+i)$-eigenbundle of this map is
$$E_\omega = \mathrm{graph}(-i\,\omega^\flat) = \{X - i\,\omega^\flat(X) \mid X \in T_{\mathbb{C}}M\},$$
where here $\omega^\flat$ denotes the complex linear extension of the original map.
This is an almost GC structure on M, and it is a GC structure if and only if $d\omega = 0$, i.e. if and only if ω is a ***symplectic structure*** on M. Since $\rho(E) = T_{\mathbb{C}}M$,
$$\mathrm{type}_x(E_\omega) = 0$$
for all $x \in M$.
(2) Let $I\colon TM \to TM$ be an ***almost complex structure*** on M, i.e. a bundle map such that $I^2 = -\mathrm{id}$. Define the map $\mathcal{J}_I\colon \mathbb{T}M \to \mathbb{T}M$ by
$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix},$$
where $I^*\colon T^*M \to T^*M$ is the dual of I. Let $T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$ be the eigenbundle decomposition of $T_{\mathbb{C}M}$ with respect to I, and $T_{\mathbb{C}}M = T^*_{1,0}M \oplus T^*_{0,1}M$ be the dual decomposition. The $(+i)$-eigenbundle of $\mathcal{J}_I$ is
$$E_I = T_{0,1}M \oplus T^*_{1,0}M.$$

This is an almost GC structure on M, and it is a GC structure if and only if I is integrable, i.e. if and only if I is a ***complex structure*** on M. Since

$\rho(E) = T_{\mathbb{C}}M,$

$$\operatorname{type}_x(E_\omega) = 2n - n = n$$

for all $x \in M$.

Cool Fact 2.5. Let M be a manifold with GC structure $\mathcal{J}\colon \mathbb{T}M \to \mathbb{T}M$, and let $\rho\colon TM \oplus T^*M \to TM$ be the natural projection. Set

$$\Pi := \rho \circ (\mathcal{J}|_{T^*M}) : T^*M \to TM.$$

Then Π is a Poisson structure on M.

The map Π is skew-symmetric for the following reason. Write $\mathcal{J}$ in terms of its tangent and cotangent coordinates, $\mathcal{J} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and note that $B = \Pi$. It is not hard to show that the adjoint of $\mathcal{J}$ with respect to the natural metric on $\mathbb{T}M$ is $\begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix}$. Since $\mathcal{J}$ is orthogonal, this also represents $\mathcal{J}^{-1}$. Because $\mathcal{J}^2 = -\mathrm{id}$, we also have $\mathcal{J}^{-1} = -\mathcal{J}$. This implies, among other things, that $B^* = -B$.

Note that if $\mathcal{J} = \mathcal{J}_\omega$ for a symplectic form ω on M, then $\Pi = -\omega^\sharp$ is the Poisson structure on M induced by the symplectic structure, (up to $\pm$).

Definition 2.6. Suppose $j\colon S \hookrightarrow M$ is a submanifold of a manifold M with GC structure $E \subset \mathbb{T}_{\mathbb{C}}M$. For each $x \in S$, let

$$E_S|_x := \{w \in \mathbb{T}_{\mathbb{C},x}S \mid \exists w' \in E_x \text{ such that } w \sim_j w'\},$$

and put $E_S = \bigcup_{x\in S} E_S|_x$. Each $E_S|_x$ is a complex linear Dirac structure on $\mathbb{T}_{\mathbb{C},x}S$, but the total space is not in general a smooth bundle; however, if it is a smooth bundle then E_S is a GC structure on S. In this case, we call (S, E_S) a ***generalized complex submanifold*** of (M, E).

3. Generalized complex Hamiltonian actions

Definition 3.1. Let M be a manifold with GC structure given equivalently by $\mathcal{J}\colon TM \to TM$ and $E \subset \mathbb{T}_{\mathbb{C}}M$. Let G be a Lie group acting smoothly on M. This induces an action of G on the Pontryagin bundle $\mathbb{T}M$ of M by

$$g \mapsto \big(g_*, (g^{-1})^*\big)$$

for $g \in G$. We say that G ***acts by symmetries*** of the GC structure if the action $G \curvearrowright \mathbb{T}M$ commutes with $\mathcal{J}$, i.e. if the diagram

$$\begin{array}{ccc} \mathbb{T}M & \xrightarrow{\mathcal{J}} & \mathbb{T}M \\ {\scriptstyle (g_*,(g^{-1})^*)}\big\downarrow & & \big\downarrow{\scriptstyle (g_*,(g^{-1})^*)} \\ \mathbb{T}M & \xrightarrow[\mathcal{J}]{} & \mathbb{T}M \end{array}$$

commutes for all $g \in G$. This is equivalent to requiring that the complex Dirac structure E be stable under the complex linear extension of $G \curvearrowright \mathbb{T}M$ to an action $G \curvearrowright \mathbb{T}_{\mathbb{C}}M$.

Example 3.2. Let G be a Lie group acting smoothly on a manifold M. If ω is a symplectic structure on M, then the action preserves ω if and only if the action is by symmetries of $\mathcal{J}_\omega$. The same is true if we replace ω with a complex structure I on M.

Definition 3.3. Let M be a manifold with a GC structure given equivalently by $\mathcal{J}\colon \mathbb{T}M \to \mathbb{T}M$ and $E \subset \mathbb{T}_{\mathbb{C}}M$, and let G be a Lie group acting on M by symmetries of this GC structure. This action is ***generalized Hamiltonian*** if there exists a G-equivariant map $\mu\colon M \to \mathfrak{g}^*$ such that, for all $\xi \in M$,

$$\xi_M = -\mathcal{J}(d\mu^\xi) \quad \text{or equivalently} \quad \xi_M - i\, d\mu^\xi \in E.$$

Here $\mu^\xi\colon M \to \mathbb{R}$ is the smooth function defined by $\mu^\xi(x) := \langle \mu(x), \xi \rangle$ for all $x \in M$. The map μ is called a ***moment map*** for the action of G on $(M, \mathcal{J})$.

Example 3.4.

(1) Let (M, ω) be a symplectic manifold, and let G be a Lie group acting on (M, ω) in a Hamiltonian fashion with moment map $\Phi\colon M \to \mathfrak{g}^*$. Recall that this means the G-action is symplectic, the map Φ is G-equivariant, and for all $\xi \in \mathfrak{g}$ we have

$$d\Phi^\xi = \omega(\xi_M, \cdot).$$

Let $\mathcal{J}_\omega$ be the GC structure on M induced by ω. The action of G on $(M, \mathcal{J}_\omega)$ is generalized Hamiltonian, and Φ is a generalized moment map. To see why, observe that for all $\xi \in \mathfrak{g}$

$$\begin{aligned}\xi_M - i\, d\mu^\xi \in E_\omega = \operatorname{graph}\left(-i\,\omega^\flat\right) &\iff -i\,\omega^\flat(\xi_M) = -i\, d\mu^\xi \\ &\iff \omega^\flat(\xi_M) = d\mu^\xi \\ &\iff \omega(\xi_M, \cdot) = d\mu^\xi.\end{aligned}$$

(2) Let a Lie group G act on a complex manifold (M, I) preserving I. This action is generalized Hamiltonian with respect to $\mathcal{J}_I$ if and only if it is trivial, because $\rho(E_I) \subset T_{\mathbb{C}}M$ contains no non-trivial real vectors. If $\mu\colon M \to \mathfrak{g}^*$ is a generalized moment map, then $\left(\xi_M - i\, d\mu^\xi\right)\big|_x \in E_I|_x$, and hence $\xi_M|_x$, for all $x \in M$, $\xi \in \mathfrak{g}$.

(3) As noted in [4, page 205], a generalized moment map $\mu\colon M \to \mathfrak{g}^*$ is also a *Poisson moment map* for the G-action with respect to the Poisson structure $\Pi := \rho \circ (\mathcal{J}|_{T^*M})$ on M.

4. Generalized complex reduction

Theorem 4.1 ([4]). *Let a compact Lie group G act on a GC manifold (M, E) in a Hamiltonian fashion with moment map $\mu\colon M \to \mathfrak{g}^*$. Let $Z = \mu^{-1}(0)$ and suppose G acts freely on Z.*

(1) *(Z, E_Z) is a GC submanifold of (M, E).*
(2) *The quotient space $M_0 := Z/G$ inherits a natural GC structure $E_0 \subset \mathbb{T}_{\mathbb{C}}M_0$.*
(3) *The quotient projection $p\colon Z \to M_0$ is a Dirac map with respect to E_Z and E_0.*
(4) *For all $x \in Z$ we have*

$$\operatorname{type}_{p(x)}(E_0) = \operatorname{type}_x(E).$$

Sketch of proof of (2). For each $x \in M$, define

$$\mathfrak{g}_M|_x := \{\, \xi_M|_x \mid \xi \in \mathfrak{g}\} \quad \text{and} \quad d\mu|_x := \{\, d\mu^\xi\big|_x \mid \xi \in \mathfrak{g}\}.$$

The total spaces $\mathfrak{g}_M$ and $d\mu$ are subspaces of TM and T^*M, respectively, but are not generally linear subbundles because they may not have constant rank. However, note that for all $x \in Z$ we have

$$T_xZ = (\, d\mu|_x)^\circ \quad \text{and} \quad T_{p(x)}(Z/G) \cong \frac{(\, d\mu|_x)^\circ}{\mathfrak{g}_M|_x},$$

where the superscript $\circ$ denotes the annihilator.

Now, fix $x \in Z$. We will construct a linear GC structure $\mathcal{J}_0|_x$ on $\mathbb{T}_{p(x)}(Z/G)$. Put $P = (\mathfrak{g}_M \oplus d\mu)|_x$. Since P is a $\mathcal{J}$-invariant subspace of $\mathbb{T}_x M$ and $\mathcal{J}$ is orthogonal, we can restrict $\mathcal{J}$ to a map $\overline{\mathcal{J}}\colon P^\perp \to P^\perp$, where the superscript $\perp$ denotes the perpendicular space with respect to the natural metric on $\mathbb{T}_x M$. Since P is also isotropic, we can take quotients by P to obtain a map $\widetilde{\mathcal{J}}\colon P^\perp/P \to P^\perp/P$. Note that both $\overline{\mathcal{J}}$ and $\widetilde{\mathcal{J}}$ square to negative the identity.

Let $V = T_x M$. Observe that $P = (\mathfrak{g}_M \oplus d\mu)|_x = (P \cap V) \oplus (P \cap V^*)$ and $P^\perp = (P \cap V^*)^\circ \oplus (P \cap V)^\circ$, so

$$\frac{P^\perp}{P} = \frac{(P \cap V^*)^\circ \oplus (P \cap V)^\circ}{(P \cap V) \oplus (P \cap V^*)} \cong \frac{(P \cap V^*)^\circ}{P \cap V} \oplus \frac{(P \cap V)^\circ}{P \cap V^*}.$$

One can show that the spaces $\frac{(P\cap V)^\circ}{P\cap V^*}$ and $\left(\frac{(P\cap V^*)^\circ}{P\cap V}\right)^*$ are naturally isomorphic. Therefore

$$\begin{aligned}\frac{P^\perp}{P} &\cong \frac{(P \cap V^*)^\circ}{P \cap V} \oplus \left(\frac{(P \cap V^*)^\circ}{P \cap V}\right)^* \\ &= \frac{(d\mu|_x)^\circ}{\mathfrak{g}_M|_x} \oplus \left(\frac{(d\mu|_x)^\circ}{\mathfrak{g}_M|_x}\right)^* \\ &\cong T_{p(x)}(Z/G) \oplus T^*_{p(x)}(Z/G) \\ &= \mathbb{T}_{p(x)}(Z/G).\end{aligned}$$

Therefore the complex structure $\widetilde{\mathcal{J}}$ on $P^\perp/P$ induces a complex structure $\mathcal{J}_0|_{p(x)}$ on $\mathbb{T}_{p(x)}(Z/G) = \mathbb{T}_{p(x)} M_0$. It remains to check that this varies smoothly with $x \in Z$ and that its $(+i)$-eigenbundle is closed under the Courant bracket. These details can be found in [4]. ☐

Remark 4.2.

(1) As noted in [4], in the context of the hypotheses of Theorem 4.1 if the GC structure and moment map come from a *symplectic* structure and moment map, then the GC structure on the quotient is exactly the one induced by the symplectic structure on the quotient.
(2) From Theorem 4.1, one can prove that if $a \in \mathfrak{g}$ is a value of μ such that G acts freely on $\mu^{-1}(\mathrm{Coad}_G(a))$, then the quotient $M_a = \mu^{-1}(\mathrm{Coad}_G(a))/G$ inherits a natural GC structure $\mathcal{J}_a$. This is accomplished by using the GC version of the "shifting trick", as described in [4].

References

[1] Theodore James Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319(2):631–661, 1990.
[2] Marco Gualtieri. *Generalized complex structures*. PhD thesis, University of Oxford, 2003.
[3] Nigel Hitchin. Generalized Calabi-Yau manifolds. *Q. J. Math.*, 54(3):281–308, 2003.
[4] Yi Lin and Susan Tolman. Symmetries in generalized Kähler geometry. *Comm. Math. Phys.*, 268(1):199–222, 2006.

Department of Mathematics, Cornell University, Ithaca, NY 14850-4201

E-mail address: goldberg@math.cornell.edu

URL: http://www.math.cornell.edu/~goldberg