Groups, groupoids, and symmetry

Student Seminar
Union College
Schenectady, New York

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Abstract

The symmetries of an object can be described as those transformations of the object that preserve its essential properties. This leads to the mantra, "Symmetries are groups." However, in some situations this mantra is incomplete, as groups cannot always capture every quality that we would clearly recognize as being some kind of symmetry. By going from groups to groupoids, we obtain a more complete way of describing symmetry, both global and local.

I will discuss some examples of symmetry in the plane, and use them to motivate the definitions of groups and groupoids. I will also provide examples of objects whose symmetry groups are small and uninformative, but whose symmetry groupoids are much richer.

This talk should be accessible to anyone who is familiar with planar geometry and basic function notation.

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Answer: An abelian **grape**.

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Outline

What is symmetry?

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A typical answer:

A **symmetry** of an object is a *transformation* of the object that preserves its essential *properties*.

Symmetries of the plane

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Every planar isometry is a

- *translation*,
- *rotation*,
- *reflection*, or
- *glide reflection*.

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3. Let T = an equilateral triangle. Sym T consists of
    - identity map,
    - 120°, 240° rotations about center of T,
    - reflections in altitude lines of T.

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3. All vertices in an equilateral triangle are "the same" but *most edge points are "different"*.

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$$f \in \mathsf{Sym}(P) \implies f^{-1} \in \mathsf{Sym}(P).$$

- *The composition of symmetries is associative.*

$$f, g, h \in \mathsf{Sym}(P) \implies (h \circ g) \circ f = h \circ (g \circ f).$$

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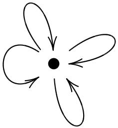
- *Associativity:* for all $x, y, z \in G$,
$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

A group as loops



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A set of loops from the same vertex, and a way of multiplying loops together, satisfying:

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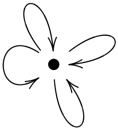

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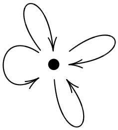

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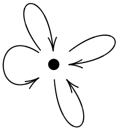

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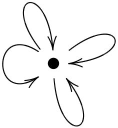

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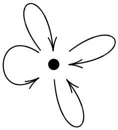

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vertex = **object**
loops = **transformations** of the object

From groups to groupoids

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When multiplication does happen, it satisfies the group axioms:
identities, **inverses**, and **associativity**.

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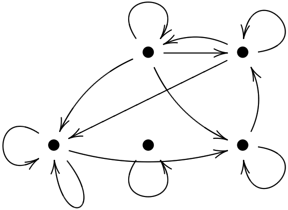

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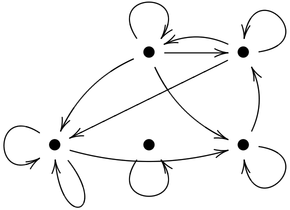

Definition

A **groupoid** $\mathbb{G}$ consists of:

- a set $\mathbb{G}_0$ of *objects*,
- a set $\mathbb{G}_1$ of *arrows* between objects, and
- a way of composing *certain arrows*

all satisfying the following properties.

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Definition, continued

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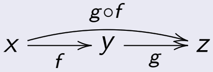

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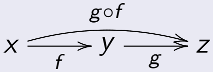

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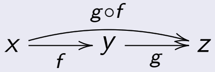

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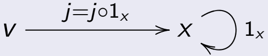

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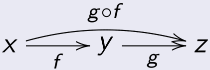

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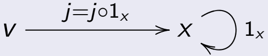

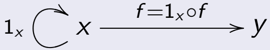

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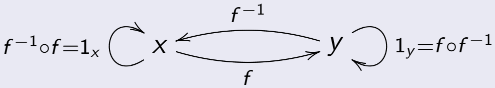

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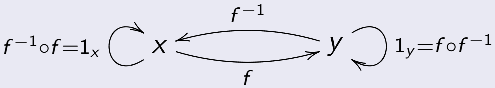

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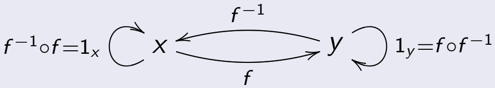

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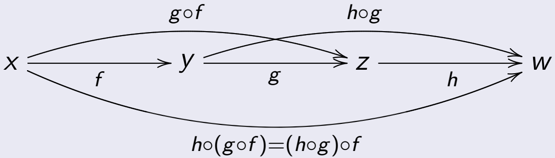

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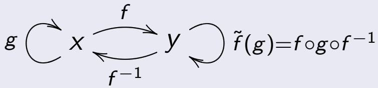

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- arrows of $\mathbb{G}$ $\rightsquigarrow$ induced maps $\rightsquigarrow$ **stems between grapes**

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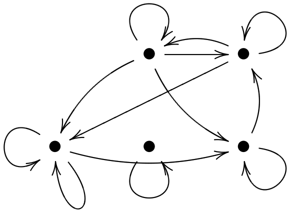

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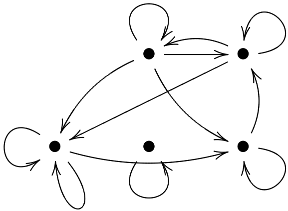
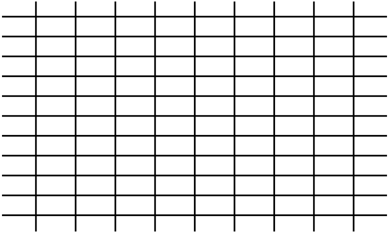


A tiled plane



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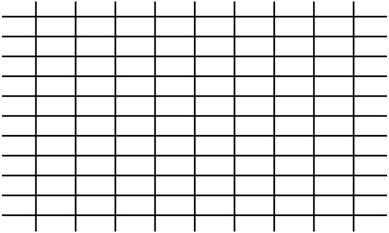
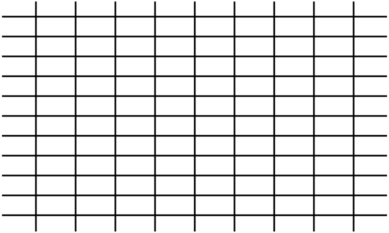


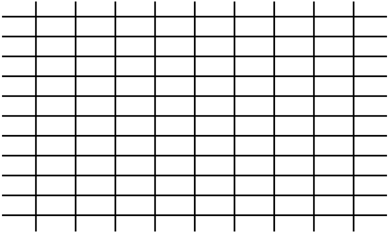
Tile the plane with rectangles.

A tiled plane



Tile the plane with rectangles. Let $X =$ the **grout** of the tiling.

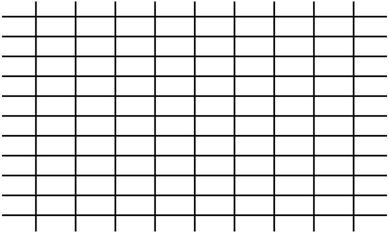
A tiled plane



Tile the plane with rectangles. Let X = the **grout** of the tiling.

Sym(X) consists of:

A tiled plane

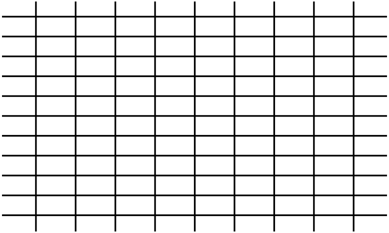


Tile the plane with rectangles. Let X = the **grout** of the tiling.

Sym(X) consists of:

- **Reflections** across vertical and horizontal lines of the grout, and those through rectangle midpoints.

A tiled plane

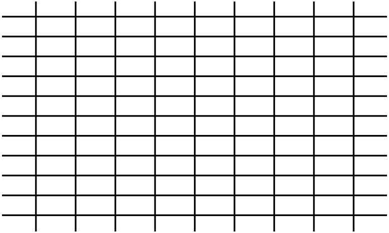


Tile the plane with rectangles. Let $X =$ the **grout** of the tiling.

Sym(X) consists of:

- **Reflections** across vertical and horizontal lines of the grout, and those through rectangle midpoints.
- **Translations** by the vectors between any two "$+$" points of the grout.

A tiled plane



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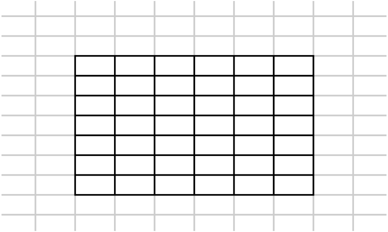
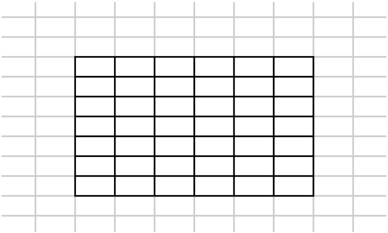
Sym(X) consists of:

- **Reflections** across vertical and horizontal lines of the grout, and those through rectangle midpoints.
- **Translations** by the vectors between any two "+" points of the grout.
- Any **combination** of the above.

A tiled floor

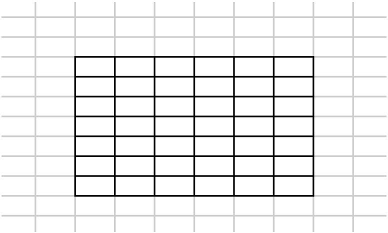


A tiled floor



Consider a rectangular tiled room, R.

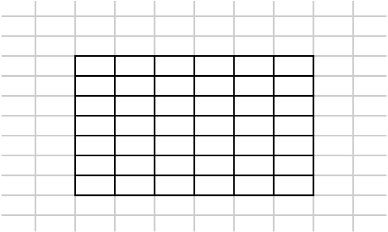
A tiled floor



Consider a rectangular tiled room, R.

$\mathrm{Sym}(R)$ has only 4 elements!

A tiled floor

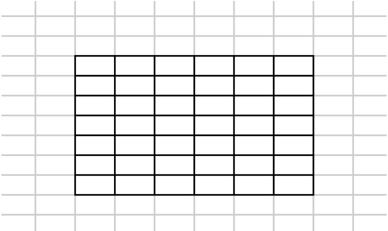


Consider a rectangular tiled room, R.

Sym(R) has only 4 elements!

- Identity.

A tiled floor

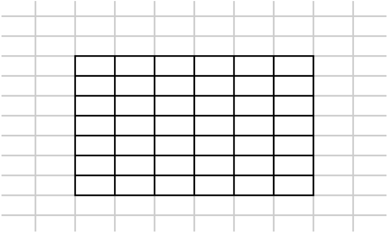


Consider a rectangular tiled room, R.

Sym(R) has only 4 elements!

- Identity.
- Reflection over horizontal line through R's midpoint.

A tiled floor



Consider a rectangular tiled room, R.

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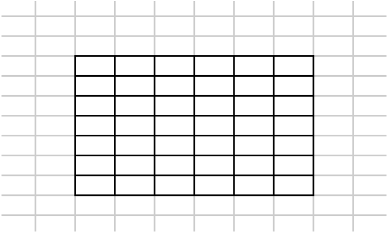
- Reflection over horizontal line through R's midpoint.
- Reflection over vertical line through R's midpoint.

A tiled floor

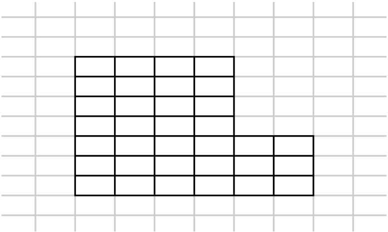


Consider a rectangular tiled room, R.

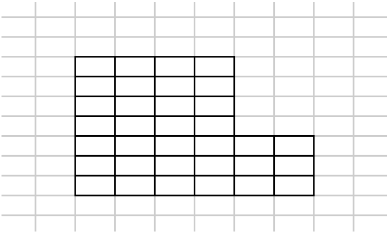
Sym(R) has only 4 elements!

- Identity.
- Reflection over horizontal line through R's midpoint.
- Reflection over vertical line through R's midpoint.
- 180° rotation about R's midpoint.

Another tiled floor

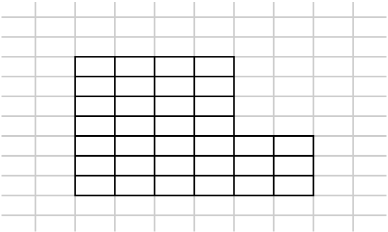


Another tiled floor



Things are even worse for this “L”-shaped room.

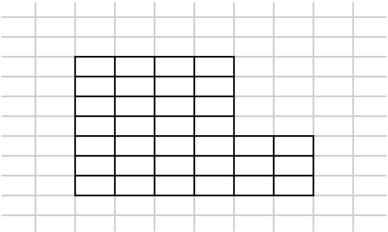
Another tiled floor



Things are even worse for this "L"-shaped room.

Its symmetry group is **just the identity!**

Another tiled floor

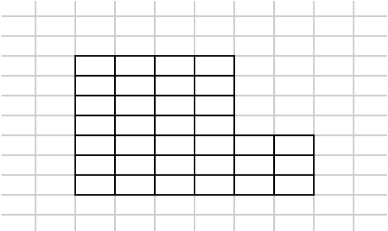


Things are even worse for this “L”-shaped room.

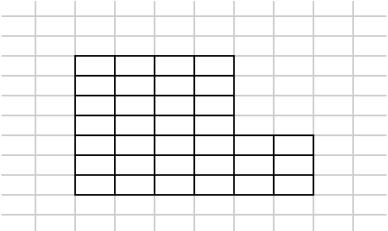
Its symmetry group is **just the identity!**

We’re missing some symmetry!

Symmetry groupoids

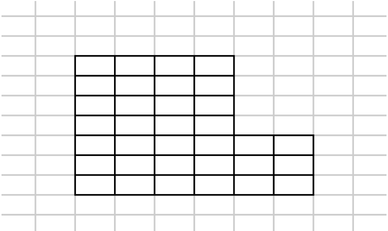


Symmetry groupoids



$\mathbb{G}(L)$, the symmetry groupoid of L

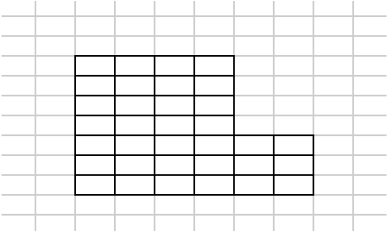
Symmetry groupoids



$\mathbb{G}(L)$, the **symmetry groupoid** of L

- Objects = points in the room L.

Symmetry groupoids



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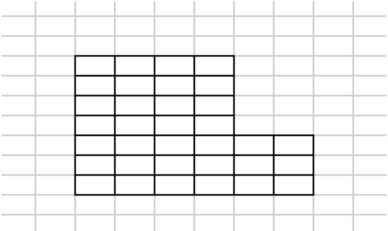

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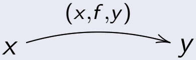
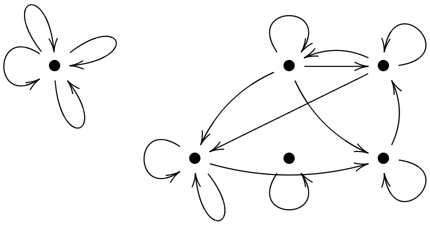


Big arrows and small arrows



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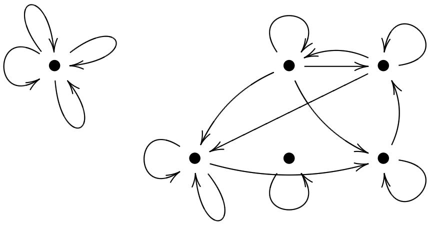
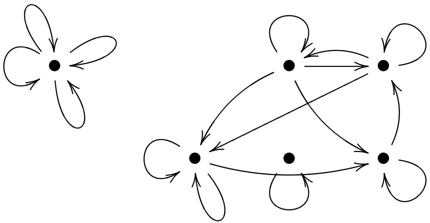


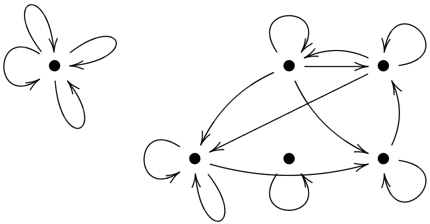
Symmetry groups consist of transformations

Big arrows and small arrows



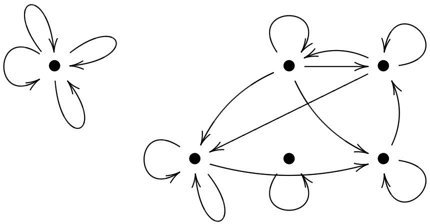
Symmetry groups consist of transformations = **big arrows**.

Big arrows and small arrows



Symmetry groups consist of transformations = **big arrows**.
They move points *all at the same time*.

Big arrows and small arrows



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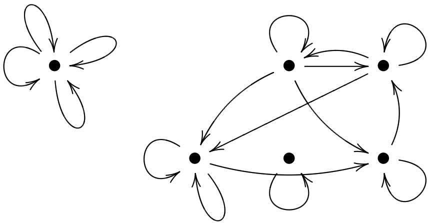
We form **symmetry groupoids** by breaking each big arrow into **small arrows**.

Big arrows and small arrows



Symmetry groups consist of transformations = **big arrows**.
They move points *all at the same time*.

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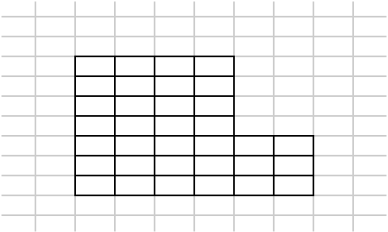
They move points *one at a time*.

Same or different?



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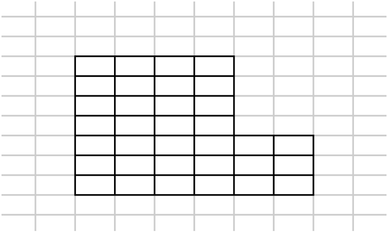
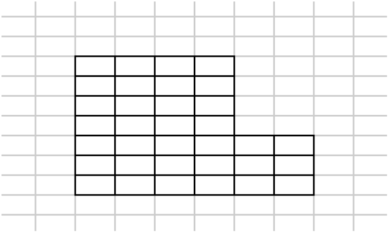


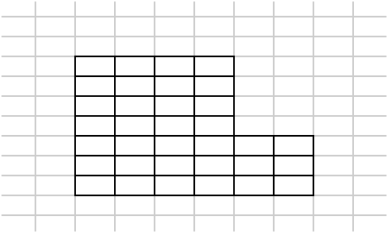
- *In terms of symmetry groupoids,*

Same or different?



- *In terms of symmetry groupoids*, $x, y \in L$ are "the same" if there is an arrow in $\mathbb{G}(L)$ from x to y.

Same or different?



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- In this example, two points are "the same" if they are similarly or symmetrically placed within their tiles.

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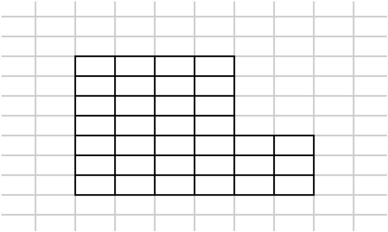

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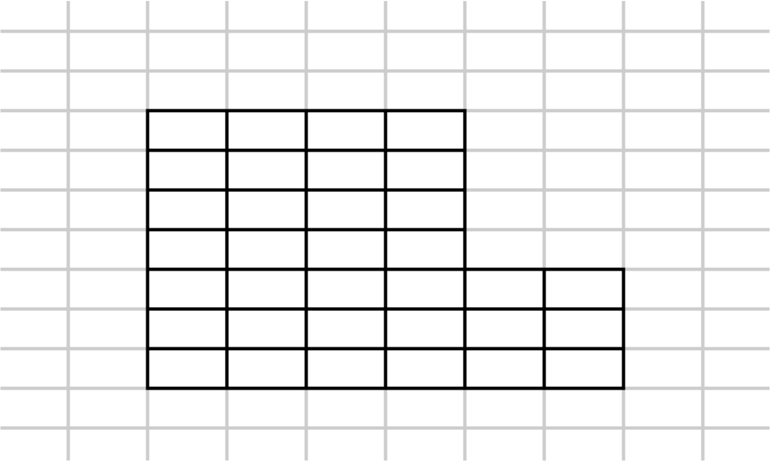
- Locally, there are even more symmetries!

Local symmetries



Local symmetries

Local symmetry types of points in L

Local symmetries

Local symmetry types of points in L

1. Interior tile points

Local symmetries

Local symmetry types of points in L

1. Interior tile points
2. Interior edge points

Local symmetries

Local symmetry types of points in L

1. Interior tile points
2. Interior edge points
3. Interior "$+$" points

Local symmetries

Local symmetry types of points in L

1. Interior tile points
2. Interior edge points
3. Interior "+" points
4. Boundary edge points

Local symmetries

Local symmetry types of points in L

1. Interior tile points
2. Interior edge points
3. Interior "+" points
4. Boundary edge points
5. Boundary "T" points

Local symmetries

Local symmetry types of points in L

1. Interior tile points
2. Interior edge points
3. Interior "+" points
4. Boundary edge points
5. Boundary "T" points
6. Acute boundary corner points

Local symmetries

Local symmetry types of points in L

1. Interior tile points
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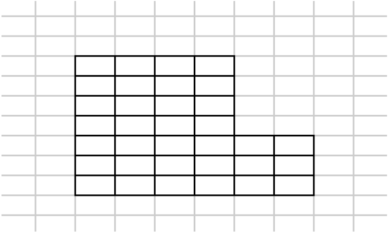
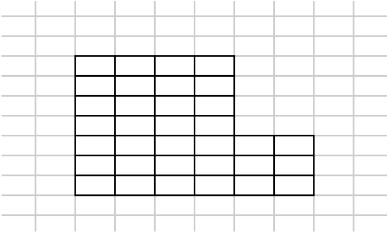
4. Boundary edge points
5. Boundary "T" points
6. Acute boundary corner points
7. Obtuse boundary corner points

Groupoids of local symmetries

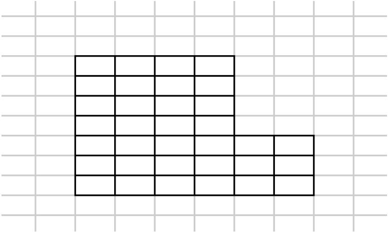


Groupoids of local symmetries



$\mathbb{G}(L)_{\text{loc}}$, the **local symmetry groupoid** of L

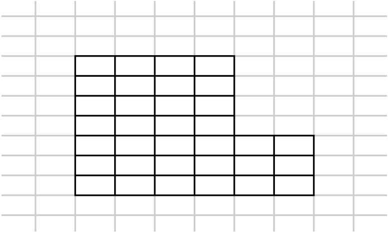
Groupoids of local symmetries



$\mathbb{G}(L)_{\text{loc}}$, the local symmetry groupoid of L

Objects = points in L.

Groupoids of local symmetries



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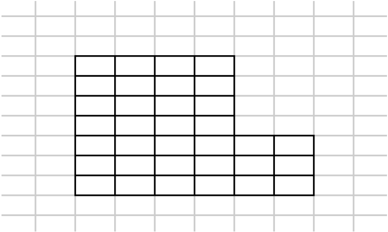
Groupoids of local symmetries



$\mathbb{G}(L)_{\text{loc}}$, the **local symmetry groupoid** of L

Objects = points in L.
Arrows = triples $(x, f, y) \in L \times \mathrm{Sym}(\mathbb{R}^2) \times L$ such that $f(x) = y$, and *locally* f preserves

- the outside of the room,
- the interior of the room tiles,
- the grout in the room.

Summary

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- **Symmetries** are ways of transforming an object while preserving its essential features.

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Summary

- **Symmetries** are ways of transforming an object while preserving its essential features.
- **Groups** are the natural algebraic structures formed by collections of *global* transformations.
- **Groupoids** are algebraic structures that can be viewed as a bunch of groups.
- When we break global transformations into point transformations, we go from symmetry **groups** to symmetry **groupoids**.
- *Groupoids allow us to capture a wider variety of symmetry phenomenon than can be captured by groups alone.*

THE END

Thank you for listening.

THE END

Thank you for listening.

And happy Earth Day!

Alan Weinstein
Groupoids: Unifying Internal and External Symmetry
Notices Amer. Math. Soc. 43 (1996), 744–752
arXiv:math/9602220