A convexity theorem for the real part of a Borel invariant subvariety

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Suppose:

\((M, \omega)\) is a compact and connected Kähler manifold, and \([\omega]\) is an \textit{integral} cohomology class.

Then:

- \(\exists\) holomorphic line bundle \(L \to M\) with Chern class \([\omega]\).
- \(\exists\) Hermitian metric on \(L\) with compatible connection \(\nabla\) whose curvature is \(\frac{1}{2\pi i} \omega\).
- \(M\) is a \textit{complex projective variety} — (Kodaira Embedding Theorem).
Suppose also:

$G$ is a compact and connected Lie group with complexification $G_{\mathbb{C}}$, and

$$G, G_{\mathbb{C}} \circlearrowright (M, L)$$

by holomorphic bundle automorphisms, preserving $\omega$ and $\nabla$.

Then:

- These actions are *algebraic* — (Serre’s GAGA).
- $\exists$ canonical $G$-equivariant *moment map*

$$\Phi : M \to g^*,$$

so $G \circlearrowright (M, \omega)$ is a *Hamiltonian action* — (Kostant).
Choose in a compatible way:

\[ T \subset G \] maximal torus

\[ t^*_+ \subset t^* \] closed positive Weyl chamber

\[ \Lambda \subset t^* \] weight lattice

\[ \Lambda_Q \subset t^* \] rational points, \( \Lambda_Q := \Lambda \otimes \mathbb{Z} \mathbb{Q} \)

\[ B \subset G_\mathbb{C} \] Borel subgroup corresponding to \( t^*_+ \)

Example:

\[ G = SU(n), \]
\[ G_\mathbb{C} = GL(n; \mathbb{C}), \]
\[ T = \{ \text{diagonal elements of } G \}, \]
\[ B = \{ \text{upper triangular elements of } G_\mathbb{C} \} \]
For each $r \geq 0$, let $\Gamma(M, L^\otimes r) = \text{holomorphic global sections of } L^\otimes r \to M$.

$$G \triangleright (M, L) \implies G \triangleright (M, L^\otimes r), \forall r \geq 0$$

$$\implies G \triangleright \Gamma(M, L^\otimes r), \forall r \geq 0.$$

So:

Each $\Gamma(M, L^\otimes r)$ is a finite-dimensional $G$-representation space, and has a \textit{weight space decomposition}

$$\Gamma(M, L^\otimes r) = \bigoplus_{\lambda \in \Lambda} \Gamma(M, L^\otimes r)_\lambda.$$
Definition:

Let $\mathcal{A} \subset M$. The \textit{highest weight space} $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$ consists of $\lambda \in \Lambda \otimes \mathbb{Q}$ such that:

- $\exists r > 0$ with $r\lambda \in \Lambda_+$,
- $r\lambda$ is a highest weight for the $G$-representation space $\Gamma(M, L \otimes r)$, and
- $\exists$ an eigensection $s \in \Gamma(M, L \otimes r)_{r\lambda}$ with $s|_{\mathcal{A}} \not\equiv 0$.

Note:

$\mathcal{C}(\mathcal{A}) \subset t^*_+$. 

Fact:

Let $X \subset M$ be a closed, irreducible, and $G_{\mathbb{C}}$-invariant subvariety. Then $\mathcal{C}(X)$ is a \textit{convex polytope} in the $\mathbb{Q}$-vector space $\Lambda \otimes \mathbb{Q}$.
Put $\Delta(X) := \Phi(X) \cap t^*_+$. 

**Theorem:** (Brion, 1986) 

If $X \subset M$ is a closed, irreducible, $G_C$-invariant subvariety, then 

$$\mathcal{C}(X) = \Delta(X) \cap \Lambda_Q \quad \text{and} \quad \overline{\mathcal{C}(X)} = \Delta(X).$$

Hence $\Delta(X)$ is a rational convex polytope. 

**Theorem:** (Guillemin–Sjamaar, 2006) 

Brion’s theorem still holds even if $X$ is only $B$-invariant.
Suppose:

- $\tau \looparrowright M$ and $\sigma \looparrowright G_\mathbb{C}$ are anti-holomorphic involutions.
- $\tau \looparrowright (M, \omega)$ is anti-symplectic.
- $\sigma$ preserves $G$ and $T$.

$$\sigma \looparrowright G \implies \sigma \looparrowright g, g^*$$

Suppose further:

- **Distribution**: $\forall g \in G, x \in M$
  $$\tau(g \cdot x) = \sigma(g) \cdot \tau(x).$$

- **Anti-equivariance**: $\forall x \in M,$
  $$\Phi(\tau(x)) = -\sigma(\Phi(x)).$$
\[ \sigma, \tau \quad \leftrightarrow \quad \text{“complex conjugation”} \]

\[ M^\tau, G^\sigma \quad \leftrightarrow \quad \text{“real parts” of } M, G. \]

**Observe:**

\[ x \in M^\tau \implies \Phi(x) = \Phi(\tau(x)) = -\sigma(\Phi(x)) \]

\[ \implies \sigma(\Phi(x)) = -\Phi(x) \]

\[ \implies \Phi(x) \in g^*_{-1}. \]

**Thus:**

\[ \Phi(M^\tau) \subset g^*_{-1}. \]
**Theorem:** (O’Shea–Sjamaar, 2000)

Suppose $X$ is preserved by $G_C$ and $\tau$, and $X^\tau$ contains a smooth point. Then

$$\Delta(X^\tau) = \Delta(X) \cap g_{-1}^*.$$ 

Hence $\Delta(X^\tau)$ is a convex polytope, because it is the intersection of the convex polytope $\Delta(X)$ (Brion’s Theorem) with the linear subspace $g_{-1}^*$. 

**Main Theorem of this Paper:** (G., 2008)

If $\sigma \bowtie G_C$ preserves $B$, then the O’Shea–Sjamaar Theorem still holds even if $X$ is only $B$-invariant.
Examples:

The main source of examples is the Borel–Weil Theorem.

If $G$ is compact and connected, $T \subset G$ is a maximal torus, $t^*_+ \subset t^*$ is a Weyl chamber, and $\lambda \in \Lambda \cap t^*_+$ is a dominant weight, then

$$M_\lambda := G_C/P_\lambda \approx G/\text{Stab}_G(\lambda) \approx \text{Coad}_G(\lambda)$$

is

- a compact, connected, and integral Kähler manifold,
- a Hamiltonian $G$-manifold, and
- something to which we can often apply the Main Theorem.
Summary:

- Convexity theorems.

- In many cases, the real part of the moment polytope is the moment polytope of the real part.

- In the integral Kähler case, the moment polytope can be described using geometric representation theory.
THE END

Thank you for listening.