A convexity theorem for the real part of a Borel invariant subvariety

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Suppose:

(M, ω) is a compact and connected Kähler manifold, and $[\omega]$ is an ***integral*** cohomology class.

Then:

- $\exists$ holomorphic line bundle $L \to M$ with Chern class $[\omega]$.
- $\exists$ Hermitian metric on L with compatible connection ∇ whose curvature is $\frac{1}{2\pi i}\,\omega$.
- M is a ***complex projective variety*** — (Kodaira Embedding Theorem).

Suppose also:

G is a compact and connected Lie group with complexification $G_{\mathbb{C}}$, and

$$G, G_{\mathbb{C}} \circlearrowright (M, L)$$

by holomorphic bundle automorphisms, preserving ω and ∇.

Then:

- These actions are ***algebraic*** — (Serre's GAGA).
- $\exists$ canonical G-equivariant ***moment map***

  $$\Phi \colon M \to \mathfrak{g}^*,$$

  so $G \circlearrowright (M, \omega)$ is a ***Hamiltonian action*** — (Kostant).

Choose in a compatible way:

$T \subset G$	maximal torus
$\mathfrak{t}^*_+ \subset \mathfrak{t}^*$	closed positive Weyl chamber
$\Lambda \subset \mathfrak{t}^*$	weight lattice
$\Lambda_{\mathbb{Q}} \subset \mathfrak{t}^*$	rational points, $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$
$B \subset G_{\mathbb{C}}$	Borel subgroup corresponding to $\mathfrak{t}^*_+$

Example:

$G = \mathbf{SU}(n)$,
$G_{\mathbb{C}} = \mathbf{GL}(n; \mathbb{C})$,
$T = \{$diagonal elements of $G\}$,
$B = \{$upper triangular elements of $G_{\mathbb{C}}\}$

For each $r \geq 0$, let $\Gamma(M, L^{\otimes r}) =$ holomorphic global sections of $L^{\otimes r} \to M$.

$$G \circlearrowright (M, L) \quad \Rightarrow \quad G \circlearrowright (M, L^{\otimes r}), \ \forall r \geq 0$$
$$\Rightarrow \quad G \circlearrowright \Gamma(M, L^{\otimes r}), \ \forall r \geq 0.$$

So:

Each $\Gamma(M, L^{\otimes r})$ is a finite-dimensional G-representation space, and has a ***weight space decomposition***

$$\Gamma(M, L^{\otimes r}) = \bigoplus_{\lambda \in \Lambda} \Gamma(M, L^{\otimes r})_{\lambda}.$$

Definition:

Let $A \subset M$. The ***highest weight space*** $\mathcal{C}(A)$ of A consists of $\lambda \in \Lambda_{\mathbb{Q}}$ such that:

- $\exists r > 0$ with $r\lambda \in \Lambda_+$,
- $r\lambda$ is a highest weight for the G-representation space $\Gamma(M, L^{\otimes r})$, and
- $\exists$ an eigensection $s \in \Gamma(M, L^{\otimes r})_{r\lambda}$ with
$$s|_A \not\equiv 0.$$

Note:

$$\mathcal{C}(A) \subset \mathfrak{t}^*_+.$$

Fact:

Let $X \subset M$ be a closed, irreducible, and $G_{\mathbb{C}}$-invariant subvariety. Then $\mathcal{C}(X)$ is a ***convex polytope*** in the $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}}$.

Put $\Delta(X) := \Phi(X) \cap \mathfrak{t}^*_+$.

Theorem: (Brion, 1986)

If $X \subset M$ is a closed, irreducible, $G_{\mathbb{C}}$-invariant subvariety, then

$$\mathcal{C}(X) = \Delta(X) \cap \Lambda_{\mathbb{Q}} \quad \text{and} \quad \overline{\mathcal{C}(X)} = \Delta(X).$$

Hence $\Delta(X)$ is a rational convex polytope.

Theorem: (Guillemin–Sjamaar, 2006)

Brion's theorem still holds even if X is only B-invariant.

Suppose:

- $\tau \curvearrowright M$ and $\sigma \curvearrowright G_{\mathbb{C}}$ are anti-holomorphic involutions.
- $\tau \curvearrowright (M, \omega)$ is anti-symplectic.
- σ preserves G and T.

$$\sigma \curvearrowright G \quad \rightsquigarrow \quad \sigma \curvearrowright \mathfrak{g}, \mathfrak{g}^*$$

Suppose further:

- **Distibution**: $\forall g \in G,\ x \in M$
$$\tau(g \cdot x) = \sigma(g) \cdot \tau(x).$$
- **Anti-equivariance**: $\forall x \in M,$
$$\Phi\left(\tau(x)\right) = -\sigma\left(\Phi(x)\right).$$

$$\sigma,\ \tau \quad \leftrightsquigarrow \quad \text{“complex conjugation”}$$

$$M^\tau,\ G^\sigma \quad \leftrightsquigarrow \quad \text{“real parts” of } M,\ G.$$

Observe:

$$x \in M^\tau \quad \Rightarrow \quad \Phi(x) = \Phi\left(\tau(x)\right) = -\sigma\left(\Phi(x)\right)$$

$$\Rightarrow \quad \sigma\left(\Phi(x)\right) = -\Phi(x)$$

$$\Rightarrow \quad \Phi(x) \in \mathfrak{g}^*_{-1}.$$

Thus:

$$\Phi\left(M^\tau\right) \subset \mathfrak{g}^*_{-1}.$$

Theorem: (O'Shea–Sjamaar, 2000)

Suppose X is preserved by $G_{\mathbb{C}}$ and τ, and X^{τ} contains a smooth point. Then

$$\Delta(X^{\tau}) = \Delta(X) \cap \mathfrak{g}^*_{-1}.$$

Hence $\Delta(X^{\tau})$ is a convex polytope, because it is the intersection of the convex polytope $\Delta(X)$ (Brion's Theorem) with the linear subspace $\mathfrak{g}^*_{-1}$.

Main Theorem of this Paper: (G., 2008)

If $\sigma \curvearrowright G_{\mathbb{C}}$ preserves B, then the O'Shea–Sjamaar Theorem still holds even if X is only B-invariant.

Examples:

The main source of examples is the ***Borel–Weil Theorem***.

If G is compact and connected, $T \subset G$ is a maximal torus, $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ is a Weyl chamber, and $\lambda \in \Lambda \cap \mathfrak{t}^*_+$ is a dominant weight, then

$$M_\lambda := G_{\mathbb{C}}/P_\lambda \approx G/\operatorname{Stab}_G(\lambda) \approx \operatorname{Coad}_G(\lambda)$$

is

- a compact, connected, and integral Kähler manifold,
- a Hamiltonian G-manifold, and
- something to which we can often apply the Main Theorem.

Summary:

- Convexity theorems.
- In many cases, the real part of the moment polytope is the moment polytope of the real part.
- In the integral Kähler case, the moment polytope can be described using geometric representation theory.

THE END

Thank you for listening.