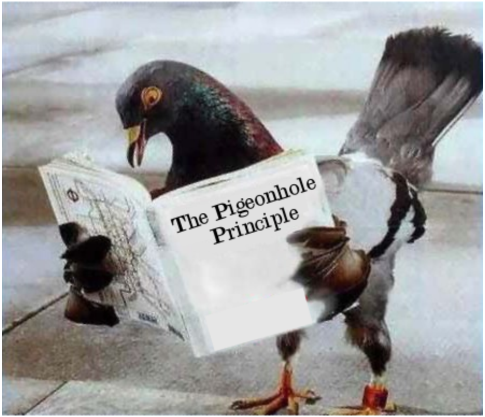
The pigeonhole principle



Marymount Manhattan College

April 14, 2010

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1. Introduction

(Not So) Magic Squares

The challenge

Fill in boxes with 1's and -1's so that **columns**, **rows**, and **diagonals** all have DIFFERENT sums.

SURPRISE!

It can't be done!

(Not So) Magic Squares

1	1
1	1

-1	-1
-1	-1

1	1
-1	-1

-1	-1
1	1

1	-1
1	-1

-1	1
-1	1

1	-1
-1	1

-1	1
1	-1

-1	1
1	1

1	-1
1	1

1	1
-1	1

1	1
1	-1

1	-1
-1	-1

-1	1
-1	-1

-1	-1
1	-1

-1	-1
-1	1

(Not So) Magic Squares

Why can't it be done?

- different sums needed $= 2$ columns $+ 2$ rows $+ 2$ diagonals $= 6$
- biggest possible sum: $1 + 1 = 2$
  smallest possible sum: $(-1) + (-1) = -2$.
- Every possible sum is between (or equal to) -2 and 2.
- BUT, only five numbers from -2 to 2.

$$\#(\text{sums needed}) > \#(\text{sums possible})$$

Therefore at least two of the sums must be the same!

This is the **Pigeonhole Principle**.

The pigeonhole principle

The principle

- If 6 pigeons have to fit into 5 pigeonholes, then some pigeonhole gets more than one pigeon.
- More generally, if $\#(\text{pigeons}) > \#(\text{pigeonholes})$, then some pigeonhole gets more than one pigeon.

Counting Argument $\rightsquigarrow$ **Combinatorics**

The pigeonhole principle

Strategy for using pigeonhole principle

- Identify the **pigeons** and **pigeonholes**.
  (Want to assign a pigeonhole for each pigeon.)
- Is $\#(\text{pigeons}) > \#(\text{pigeonholes})$?
- If YES, then some pigeonhole has to get more than one pigeon!

EXAMPLE: (Not So) Magic Squares

$$\begin{aligned} \textbf{pigeons} &= \text{different sums needed (6)} \\ \textbf{pigeonholes} &= \text{possible sums } (< 5) \end{aligned}$$

Therefore 2 (or more) sums must be the same.

What about 6×6?

			-1		
	1	-1			
1					1
		-1		-1	
					1

- different sums needed = 6 columns + 6 rows + 2 diagonals = 14
- biggest possible sum: $1+1+1+1+1+1=6$
  smallest possible sum:
  $(-1)+(-1)+(-1)+(-1)+(-1)+(-1)=-6$.

pigeons = different sums needed (14)
pigeonholes = possible sums (< 13)

Nope! (Actually doesn't work for any $n \times n$.)

2. Examples

Someone's been using my initials.

How many first/last name initials are there?

- 26 possible letters.
- $26 \times 26 = 676$ possible pairs of initials.

CLAIM: At least 2 students at **Marymount Manhattan College** have the same first/last initials.

$$\begin{aligned} \textbf{pigeons} &= \text{MMC students} \\ \textbf{pigeonholes} &= \text{possible first/last initials} \\ \#(\text{pigeons}) &\approx 2{,}100 \\ \#(\text{pigeonholes}) &= 676 \end{aligned}$$

Warning: Doesn't mean every student has an "initial twin"!

Someone's been using my initials.

How many first/middle/last name initials are there?

- 26 possible letters.
- Some people have no middle names, so include "blank" for middle initial.
- $26 \times 27 \times 26 = 18,252$ possible triples of initials.

CLAIM: At least 2 students at **Cornell University** have the same first/middle/last initials.

$$\begin{aligned} \textbf{pigeons} &= \text{CU students} \\ \textbf{pigeonholes} &= \text{possible first/middle/last initials} \\ \#(\text{pigeons}) &\approx 20{,}600 \\ \#(\text{pigeonholes}) &= 18{,}252 \end{aligned}$$

Hairs in New York City

CLAIM: At any time in New York City, there are 2 people with the same number of hairs.

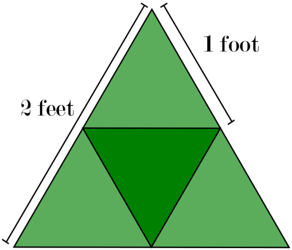
$$\begin{aligned} \textbf{pigeons} &= \text{people in New York City} \\ \textbf{pigeonholes} &= \text{possible \# of hairs} \\ \#(\text{pigeons}) &\approx 8{,}363{,}000 \\ \#(\text{pigeonholes}) &< 7{,}000{,}000 \end{aligned}$$

A triangular dartboard

Dartboard = equilateral triangle with side length of 2 feet

CLAIM: If you throw 5 darts (no misses), at least 2 will be within a foot of each other.

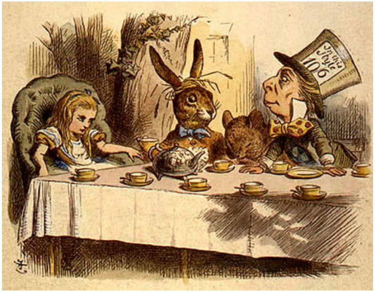
A triangular dartboard



- Divide triangle into 4 sub-triangles.
- Darts in same sub-triangle are within 1 foot of each other.

pigeons	=	darts (5)
pigeonholes	=	sub-triangles (4)

A party problem



Set-Up:

- Party with 10 people.
- Each guest counts how many guests she/he has met before.

Cool Fact:

At least 2 people will have met the same number of guests before!

A party problem

Cool Fact:

At least 2 people will have met the same number of guests before!

Why?

pigeons	=	party guests
pigeonholes	=	possible number of guests met before

- How many guests has each person met before? (0 – 9)
- 0 = met **no one** before.
  9 = met **everyone** before.
- 0 and 9 can't happen at the same party!
- number of guests met before: only nine possiblities!
  (0 – 8 or 1 – 9)

A party problem

Cool Fact:

At least 2 people will have met the same number of guests before!

pigeons	$=$	party guests (10)
pigeonholes	$=$	possible number of guests met before (9)

Birthday twins!

Question: How many people do you need to guarantee 2 of them share a birthday?

What are the odds?

So:

$$366 + 1 = 367 \text{ people} \rightsquigarrow 100\% \text{ chance of shared birthday}$$

It's amazing!

23 people	50%
57 people	99%
100 people	99.9999%
200 people	99.999999999999999999999999999%

This is called **The Birthday Problem**.
Not really Pigeonhole Principle, but still about counting things.

THE END

Thank you for listening.

For many more Pigeonhole puzzles and examples, please see the Internet.