A little taste of symplectic geometry

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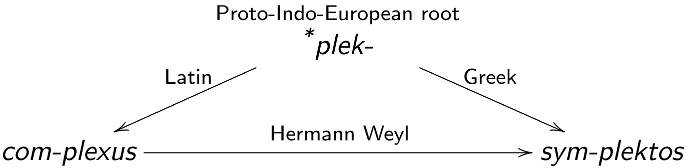
Abstract

Symplectic geometry is a rich and beautiful field in pure mathematics whose origins lie in classical physics. Specifically, symplectic spaces arose as the natural setting in which to study Hamiltonian mechanics. A symplectic structure is precisely what is needed to associate a dynamical system on the space to each energy function.

I will give a little taste of what this subject is about, focusing on symplectic structures on vector spaces and surfaces, and symmetries of these structures. This talk should be accessible to students with a background knowledge of multivariable calculus and some basic linear algebra.


BORK, BORK,
$df = \iota(X_f)\,\omega$,
BORK.
Contact and

What does "symplectic" mean?



"complex" = "symplectic" = "braided or plaited together"

Outline

1. Classical mechanics

Particles moving in 3-space

Let $\mathbb{R}^3$ = **configuration space** = set of possible positions.

$$\text{Position coordinates: } \mathbf{q} = (q_1, q_2, q_3)$$

Consider a particle with mass m moving in configuration space under a **potential** $V(\mathbf{q})$.

$$V : \text{configuration space} \longrightarrow \mathbb{R}, \text{ (potential energy)}$$

$$\text{particle's path: } \mathbf{q}(t) = \big(q_1(t), q_2(t), q_3(t)\big)$$

Newton's Second Law

Newton's Second Law

The force imparted upon the particle from the potential is equal to the product of the particle's mass and acceleration.

$$m\,\mathbf{q}''(t) = -\nabla V\big(\mathbf{q}(t)\big)$$

Particles want to decrease their potential energy, $-\nabla V$ points the way.

Hamiltonian reformulation: phase space

$$\left.\begin{array}{ll} \text{position coordinates} & \mathbf{q} = (q_1, q_2, q_3) \\ \text{momenta coordinates} & \mathbf{p} = (p_1, p_2, p_3) \end{array}\right\} \text{phase space}$$

$$\begin{array}{ccccc} \text{configuration space} & \& & \text{possible momenta} & = & \textbf{phase space} \\ \mathbb{R}^3 & \times & \mathbb{R}^3 & = & \mathbb{R}^6 \end{array}$$

$$\mathbf{p}(t) = m\,\mathbf{q}'(t)$$

$$\mathbf{q}(t) \text{ in configuration space} \quad \rightsquigarrow \quad \big(\mathbf{q}(t), \mathbf{p}(t)\big) \text{ in phase space}$$

$$m\,\mathbf{q}''(t) = -\nabla V\big(\mathbf{q}(t)\big) \quad \rightsquigarrow \quad \mathbf{p}'(t) = -\nabla V\big(\mathbf{q}(t)\big)$$

Hamiltonian reformulation: total energy function

Definition

The **Hamiltonian function** of a physical system,

$$H\colon \text{phase space} \to \mathbb{R},$$

measures the total energy of a moving particle.

$$\begin{aligned} H(\mathbf{q},\mathbf{p}) &= \text{kinetic energy} &+& \text{potential energy} \\ &= \tfrac{1}{2}\, m\, |\mathbf{v}|^2 &+& V(\mathbf{q}) \\ &= \tfrac{1}{2m}\, |\mathbf{p}|^2 &+& V(\mathbf{q}) \end{aligned}$$

Hamilton's Equations

$$\boxed{H(\mathbf{q},\mathbf{p}) = \frac{1}{2m}\left|\mathbf{p}\right|^2 + V(\mathbf{q}) = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + V(q_1, q_2, q_3)}$$

$p_i(t) = m\, q_i'(t)$:

$$q_i'(t) = \frac{1}{m}\, p_i(t) = \frac{\partial H}{\partial p_i}\big(\mathbf{q}(t), \mathbf{p}(t)\big)$$

$\mathbf{p}'(t) = -\nabla V\big(\mathbf{q}(t)\big)$:

$$p_i'(t) = -\frac{\partial V}{\partial q_i}\big(\mathbf{q}(t)\big) = -\frac{\partial H}{\partial q_i}\big(\mathbf{q}(t), \mathbf{p}(t)\big)$$

Hamilton's Equations

Hamilton's Equations

$$
\begin{aligned}
q_i'(t) &= \frac{\partial H}{\partial p_i}(\mathbf{q}(t), \mathbf{p}(t)) \\
p_i'(t) &= -\frac{\partial H}{\partial q_i}(\mathbf{q}(t), \mathbf{p}(t))
\end{aligned}
$$

- Newton's Second Law $\iff$ Hamilton's Equations
- particle's motion satisfies the **dynamics** of the physical system $\iff$ Hamilton's equations are satisfied

The Hamiltonian vector field

$q_i' = \frac{\partial H}{\partial p_i}, \; p_i' = -\frac{\partial H}{\partial q_i}$

Definition

The **Hamiltonian vector field** of H is the vector field X_H on phase space defined by

$$X_H := \left(\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \frac{\partial H}{\partial p_3}, -\frac{\partial H}{\partial q_1}, -\frac{\partial H}{\partial q_2}, -\frac{\partial H}{\partial q_3}\right)$$

(dynamics determined by the energy function H)

Hamilton's Equations, second version

$$\big(\mathbf{q}'(t), \mathbf{p}'(t)\big) = X_H\big(\mathbf{q}(t), \mathbf{p}(t)\big)$$

Key property of the Hamiltonian vector field

Claim

X_H points in directions of constant energy.

Proof. Suffices to show derivative of H in direction X_H is zero.

$$\begin{aligned} D_{X_H}(H) &= \nabla H \cdot X_H \\ &= \left(\frac{\partial H}{\partial \mathbf{q}}, \frac{\partial H}{\partial \mathbf{p}}\right) \cdot \left(\frac{\partial H}{\partial \mathbf{p}}, -\frac{\partial H}{\partial \mathbf{q}}\right) \\ &= \left(\frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}}\right) - \left(\frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}}\right) \\ &= 0. \end{aligned}$$

□

The symplectic structure

H	$\Longrightarrow$	X_H
Hamiltonian function		Hamiltonian vector field
energy		*dynamics*
	$\uparrow$	
	symplectic structure	

2. Symplectic linear algebra

Bilinear forms

Let V = real n-dimensional vector space.

Definition

A **bilinear form** is a map $\beta\colon V \times V \to \mathbb{R}$ that is linear in each variable.

$$\begin{aligned}\beta(a\,\mathbf{u} + \mathbf{v}, \mathbf{w}) &= a\,\beta(\mathbf{u}, \mathbf{w}) + \beta(\mathbf{v}, \mathbf{w}),\\ \beta(\mathbf{u}, a\,\mathbf{v} + \mathbf{w}) &= a\,\beta(\mathbf{u}, \mathbf{v}) + \beta(\mathbf{u}, \mathbf{w}).\end{aligned}$$

Given a basis for V, can represent β by an $(n \times n)$-matrix $[\beta]$:

$$\beta(\mathbf{u}, \mathbf{v}) = [\mathbf{u}]^t\,[\beta]\,[\mathbf{v}].$$

Example: dot product on $\mathbb{R}^2$:

$$(u_1, u_2) \cdot (v_1, v_2) = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2.$$

Linear symplectic structures

Definition

A **symplectic product**, a.k.a. **linear symplectic structure**, is a *skew-symmetric* and *non-degenerate* bilinear form.

skew-symmetric:

- $\beta(\mathbf{u}, \mathbf{v}) = -\beta(\mathbf{v}, \mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in V$
- $[\beta]^t = -[\beta]$

non-degenerate:

- $(\beta(\mathbf{u}, \mathbf{v}) = 0 \text{ for all } \mathbf{v} \in V) \iff (\mathbf{u} = \mathbf{0})$
- $[\beta]$ is invertible

Symplectic products versus inner products

symplectic product = skew-symmetric & non-degenerate

Compare with:

inner product = symmetric & positive definite
($\Rightarrow$ non-degenerate)
e.g. dot product on $\mathbb{R}^n$

Note: if β is an inner product, then $\sqrt{\beta(\mathbf{u}, \mathbf{u})} =$ "length" of $\mathbf{u}$,
e.g. $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$ in $\mathbb{R}^n$,
BUT if β is symplectic, then $\beta(\mathbf{u}, \mathbf{u}) = 0$ for all $\mathbf{u}$!

Symplectic $\Rightarrow$ even-dimensional

All vector spaces have *inner products*, but not all have *symplectic products*!

Claim

If $\omega\colon V \times V \to \mathbb{R}$ is symplectic, then $\dim V$ is even.

Proof. Let $n = \dim V$.

- By skew-symmetry: $[\omega]^t = -[\omega]$,

  so $\det[\omega] = \det([\omega]^t) = \det(-[\omega]) = (-1)^n \det[\omega]$,

  so $\det[\omega] = (-1)^n \det[\omega]$.

- By non-degeneracy: $[\omega]$ is invertible,

  so $\det[\omega] \neq 0$, so $(-1)^n = 1$, so n is even.

$\square$

An example

Let $V = \mathbb{R}^2$, $\mathbf{u} = (x, y)$, $\mathbf{v} = (a, b)$.

Define ω by $[\omega] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$$\begin{aligned} \omega(\mathbf{u}, \mathbf{v}) &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= xb - ay \\ &= \det \begin{bmatrix} x & a \\ y & b \end{bmatrix} \\ &= \text{oriented area of parallelogram}(\mathbf{u}, \mathbf{v}). \end{aligned}$$

The standard symplectic structure on $\mathbb{R}^{2n}$

Definition

The **standard symplectic structure** ω on $\mathbb{R}^{2n}$ is $[\omega] = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}$.

If $\mathbf{u} = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, $\mathbf{v} = (a_1, \ldots, a_n, b_1, \ldots, b_n)$, then

$$\omega(\mathbf{u}, \mathbf{v}) = (x_1 b_1 - a_1 y_1) + \cdots + (x_n b_n - a_n y_n).$$

Note:

$$[\omega]^t = \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}^t = \begin{bmatrix} 0^t & (-\mathrm{I})^t \\ \mathrm{I}^t & 0^t \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} = -[\omega],$$

$$\det[\omega] = -\det(\mathrm{I}) \cdot \det(-\mathrm{I}) = (-1)^{n+1} \neq 0.$$

The gradient and Hamiltonian vector fields

Let $f\colon \mathbb{R}^{2n} \to \mathbb{R}$ be differentiable.
Recall: the **gradient** of f is the vector field ∇f defined by

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}\right).$$

Alternatively, can be defined by the identity

$$D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{R}^{2n}.$$

Definition

The **Hamiltonian vector field** of f, a.k.a. the **symplectic gradient**, is the unique vector field X_f satisfying

$$D_{\mathbf{v}} f = \omega(X_f, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbb{R}^{2n}.$$

(X_f exists because ω is non-degenerate.)

Key property of Hamiltonian vector fields

Claim

A function $f\colon \mathbb{R}^{2n} \to \mathbb{R}$ is constant in the direction of its Hamiltonian vector field X_f.

Proof. Suffices to show derivative of f in direction X_f is zero.

$$D_{X_f}(f) = \omega\big(X_f, X_f\big) = 0,$$

since ω is skew-symmetric. □

Computing Hamiltonian vector fields

Claim

For the standard ω on $\mathbb{R}^{2n}$,

$$X_f = \left(\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}, -\frac{\partial f}{\partial x_1}, \ldots, -\frac{\partial f}{\partial x_n}\right).$$

Proof. Let $\mathbf{v} = (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$. Then

$$\begin{aligned}
\omega(X_f, \mathbf{v}) &= \begin{bmatrix} \frac{\partial f}{\partial \mathbf{y}} & -\frac{\partial f}{\partial \mathbf{x}} \end{bmatrix} \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \\
&= \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{a} + \frac{\partial f}{\partial \mathbf{y}} \cdot \mathbf{b} \\
&= \nabla f \cdot \mathbf{v} \\
&= D_{\mathbf{v}}(f).
\end{aligned}$$

□

Playing "Find the Hamiltonian!"

Let χ be a vector field on $\mathbb{R}^{2n}$.

Is χ **conservative**? $\longleftrightarrow$ Is there an f such that $\chi = \nabla f$?

Is χ **Hamiltonian**? $\longleftrightarrow$ Is there an f such that $\chi = X_f$?

$\longleftrightarrow$ Does χ represent the dynamics corresponding to some energy function?

3. Momentum & moment maps

Rotations in 3-space

Back to phase space of $\mathbb{R}^3$, with coordinates $(\mathbf{q}, \mathbf{p})$.
Consider rotations in $\mathbb{R}^3$.

$\mathbf{a} \in \mathbb{R}^3 \quad \rightsquigarrow \quad R_{\mathbf{a}}$, a rotation about axis spanned by $\mathbf{a}$ according to the right-hand rule

$\rightsquigarrow \quad \chi_{\mathbf{a}}$, a vector field on phase space, points tangent to direction of rotation

$$\chi_{\mathbf{a}}\big(\mathbf{q}, \mathbf{p}\big) = \big(\mathbf{a} \times \mathbf{q}, \mathbf{a} \times \mathbf{p}\big)$$

Specifics of the rotations

$$\mathbf{a} \in \mathbb{R}^3 \quad \rightsquigarrow \quad A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \mathfrak{so}(3, \mathbb{R})$$

$$\rightsquigarrow \quad R_{\mathbf{a}} := \exp A \in \mathrm{SO}(3, \mathbb{R})$$

$R_{\mathbf{a}}$ is rotation by angle $\cos^{-1}(\cos \|\mathbf{a}\|)$ about the axis spanned by $\mathbf{a}$ according to the right-hand rule.

Angular momentum

Angular momentum:

$$\mu\colon \text{phase space} \to \mathbb{R}^3, \qquad \mu(\mathbf{q},\mathbf{p}) := \mathbf{q} \times \mathbf{p}.$$

For the rotation given by $\mathbf{a} \in \mathbb{R}^3$:

$$\mu^{\mathbf{a}}\colon \text{phase space} \to \mathbb{R}, \qquad \mu^{\mathbf{a}}(\mathbf{q},\mathbf{p}) := (\mathbf{q} \times \mathbf{p}) \cdot \mathbf{a}.$$

Cool fact

$\mu^{\mathbf{a}}$ is the Hamiltonian function for the rotation vector field $\chi_{\mathbf{a}}$!

$$\chi_{\mathbf{a}} = X_{\mu^{\mathbf{a}}}$$

This is a prototype for **moment maps** in symplectic geometry! (winning strategy for "Find the Hamiltonian!" from symmetry vector fields)

Hermitian matrices

Definition

A complex $(n \times n)$-matrix A is **Hermitian** if

$$\overline{A^t} = A.$$

Hermitian matrices ...

- ... have real diagonal entries.
- ... are diagonalizable.
- ... have real eigenvalues.

The Schur–Horn Theorem

Fix n real numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Let $\mathcal{H}_\lambda$ = set of Hermitian matrices with eigenvalues λ (in any order), a.k.a. **isospectral set**.

Schur–Horn Theorem

Define $f : \mathcal{H}_\lambda \to \mathbb{R}^n$ by $f(A) = \text{diagonal}(A)$,

$$\begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix} \mapsto \begin{bmatrix} A_{11} \\ \vdots \\ A_{nn} \end{bmatrix}.$$

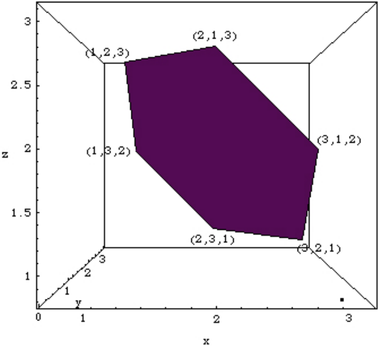
Then

- $f(\mathcal{H}_\lambda)$ is a **convex polytope**, and
- its vertices are the permutations of λ.

A Schur–Horn example

Example: $n = 3$, $\lambda = (3, 2, 1)$.

$f(\mathcal{H}_\lambda)$ lives in $\mathbb{R}^3$, but is contained in the plane $x + y + z = 6$.



The symplectic convexity theorem

Schur–Horn Theorem: originally proved by min/max argument (1950s). No symplectic stuff.

BUT IN FACT:

- $\mathcal{H}_\lambda$ is a symplectic space.
- $f : \mathcal{H}_\lambda \to \mathbb{R}^n$ is a symplectic moment map.
- Schur–Horn Theorem is a special case of the **Atiyah/Guillemin–Sternberg Theorem** (1982), about convexity of images of moment maps. (MUCH more general)

4. Winding down

Summary

- **Hamilton's Equations**: H (energy) $\rightsquigarrow$ X_H (dynamics)
- **Linear symplectic structure** $\approx$ skew-symmetric inner product.
- **Symplectic gradient**: $f \overset{\omega}{\longmapsto} X_f$
- **Moment maps** find the Hamiltonian for symmetry vector fields,
  e.g. **angular momentum**.
- Moment maps have **great properties**, e.g. convexity theorem.
- **Schur–Horn Theorem**:
  A convexity theorem example you can get your hands on.

THE END

Thank you for listening.