# A little taste of symplectic geometry

Timothy Goldberg

Thursday, October 4, 2007
Olivetti Club Talk
Cornell University


SUPERMAN
RETURNS

What is symplectic geometry?

Symplectic geometry is the study of the geometry of symplectic manifolds!

## The game plan

0. Prologue: Schur–Horn theorem (original version)

1. Symplectic vector spaces

2. Symplectic manifolds

3. Hamiltonian group actions

4. Atiyah/Guillemin–Sternberg theorem

5. Epilogue: Schur–Horn theorem (symplectic version)

# 0. Prologue

Let $\mathcal{H}(n) = \{$**Hermitian** $(n \times n)$–matrices$\}$. ($\bar{A}^T = A$)

Hermitian $\Longrightarrow$ real diagonal entries and eigenvalues.

Put

$\vec{\lambda} = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$,

$\mathcal{O}_{\vec{\lambda}} = \{A \in \mathcal{H}(n)$ with eigenvalues $\vec{\lambda}\}$ (**isospectral set**),

$f\colon \mathcal{O}_{\vec{\lambda}} \to \mathbb{R}^n$, $f(A) =$ diagonal of $A$.

**Thereom:** [Schur-Horn, mid-1950's]

$f(\mathcal{O}_{\vec{\lambda}})$ is a **convex** polytope in $\mathbb{R}^n$, the **convex hull** of vectors whose entries are $\lambda_1, \ldots, \lambda_n$ (in some order).

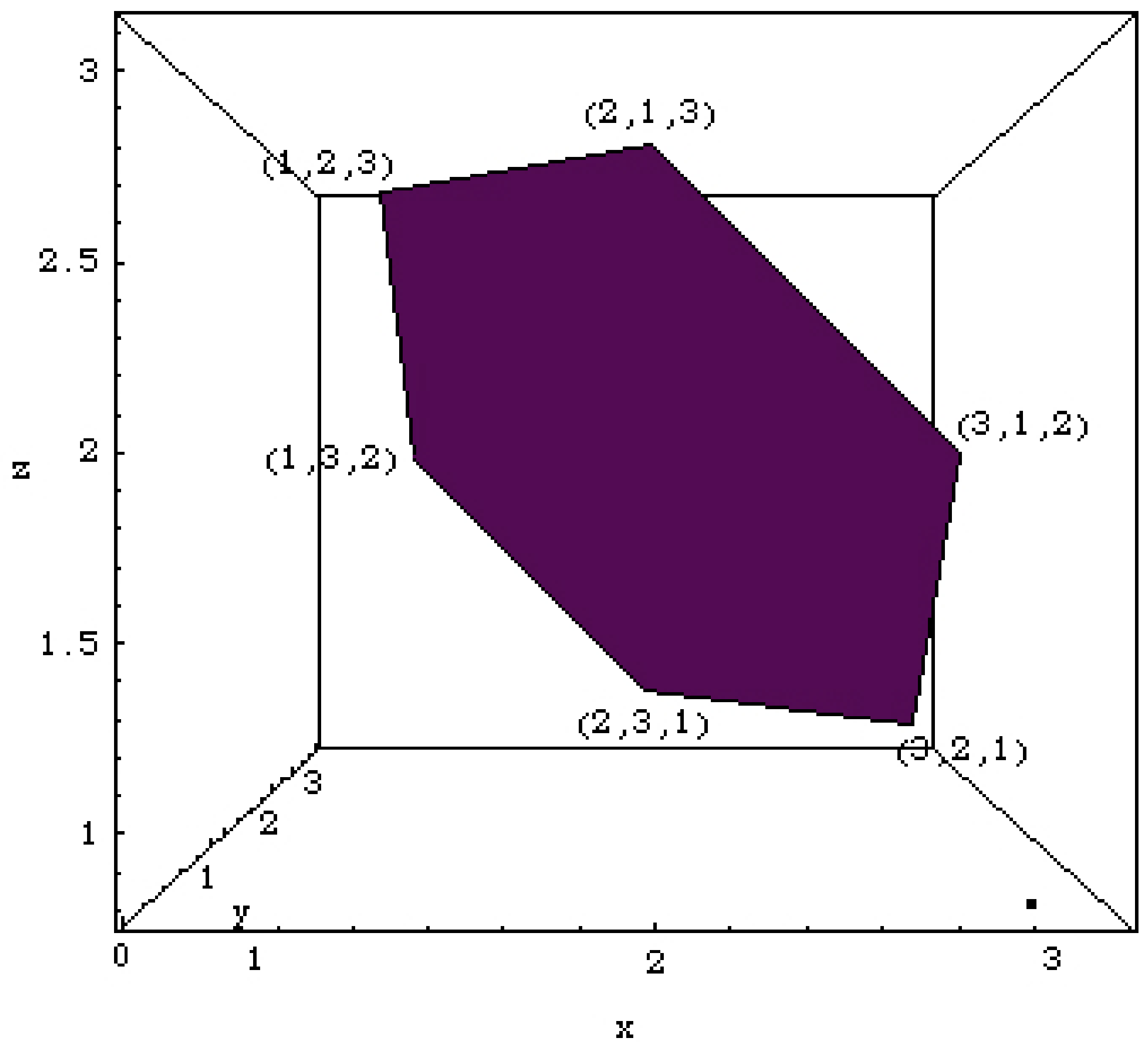
**Definition:** $C$ is **convex** if $a, b \in C \Longrightarrow \overline{ab} \subset C$.

The **convex hull** of $P$ is the smallest convex set containing $P$.

A **convex polytope** is the convex hull of a finite set of points.

**Example:** $n = 3$, $\vec{\lambda} = (3, 2, 1)$.

$f(\mathcal{O}_{\vec{\lambda}})$ lives in $\mathbb{R}^3$, but is contained in the plane $x + y + z = 6$.



# 1. Symplectic vector spaces

$V$ = finite dimensional real vector space

**Definition:** An **inner product** on $V$ is a map $g\colon V \times V \to \mathbb{R}$ with the following properties.

- $g$ is **bilinear**
- $g$ is **symmetric**
- $g$ is **positive definite**

Note: positive definite $\Longrightarrow$ **nondegenerate**.

**Example:** $V = \mathbb{R}^n$, $g$ = standard dot product

**Definition:** A **symplectic product** on $V$ is a map $\omega\colon V \times V \to \mathbb{R}$ with the following properties.

- $\omega$ is **bilinear**
- $\omega$ is **skew-symmetric**
- $\omega$ is **nondegenerate**

(Note that for all $v \in V$, $\omega(v,v) = 0$.)

A **symplectic vector space** is a vector space equipped with a symplectic product.

Every (finite dimensional) vector space has an inner product, but *not every vector space has a symplectic product!*

**Claim:** If $V$ has a symplectic product $\omega$, then $\dim V$ is even.

**Proof:** Let $A$ be the matrix of $\omega$ relative to some basis for $V$. Then

$$\det A = \det A^T = \det(-A) = (-1)^n \det A,$$

where $n = \dim V$. Since $\det A \neq 0$, $1 = (-1)^n$, so $n = \dim V$ is even.

$\mathbb{QED}$

**Example:** $V = \mathbb{R}^{2n}$, $\omega = \omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. (**standard symplectic product**)

If $n = 2$:

$$\begin{aligned} \omega(\vec{x}, \vec{y}) &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= x_1 y_2 - x_2 y_1 = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \det \begin{pmatrix} \vec{x} & \vec{y} \end{pmatrix} \\ &= \text{oriented area of the parallelogram spanned by } \vec{x}, \vec{y}. \end{aligned}$$

Thus, every even-dimensional vector space has a symplectic product, and in fact, up to a change of coordinates, *every symplectic product looks like this one!*

The **gradient** of $f\colon \mathbb{R}^n \to \mathbb{R}$ is the vector field

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

**Coordinate-free definition:** $\nabla f$ is the unique vector field such that $\forall p \in \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$,

$$\boxed{(D_{\vec{v}} f)(x) = \nabla f(x) \cdot \vec{v}.}$$

($D_{\vec{v}} f =$ **directional derivative** of $f$ in the direction $\vec{v}$.)

The **symplectic gradient** of $f$ is the unique vector field $\nabla_\omega f$ such that $\forall p \in \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$,

$$\boxed{(D_{\vec{v}} f)(x) = \omega\left(\nabla_\omega f(x), \vec{v}\right).}$$

(The uniqueness follows from nondegeneracy.)

**Example:** $V = \mathbb{R}^2$, $\omega = \omega_0 =$ standard symplectic form.

$$\nabla_\omega f = \left( -\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right)$$

Let $f(x, y) = x^2 + y^2$. Then

$$\nabla f = (2x, 2y) \quad \text{and} \quad \nabla_\omega f = (-2y, 2x).$$

$\nabla f$ is **perpendicular** to level curves of $f$, and points to **increasing** values of $f$.

$\nabla_\omega f$ is **tangent** to level curves of $f$, and points to **constant** values of $f$.

$$\left(D_{\nabla_\omega f(p)} f\right)(p) = \omega\left(\nabla_\omega f(p), \nabla_\omega f(p)\right) = 0.$$

$$\begin{array}{rcl} f & \rightsquigarrow & \text{energy function} \\ \nabla f & \rightsquigarrow & \text{points to increasing energy} \\ \nabla_\omega f & \rightsquigarrow & \text{points to stable energy} \end{array}$$

*Symplectic geometry is the natural setting for studying classical mechanics!*

**A game we can play: Find the Hamiltonian!**

Usual version

Given a vector field $X$ on $\mathbb{R}^n$, find a function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\nabla f = X.$$

Symplectic version

Given a vector field $X$ on $V$, find a function $f\colon V \to \mathbb{R}$ such that

$$\nabla_\omega f = X.$$

Classical mechanics interpretation

The vector field represents a system of moving particles (Hamiltonian system). We want to find an **energy function** (Hamiltonian) for this system.

We are basically trying to solve **Hamilton's equations.**

# 2. Symplectic manifolds

**Definition:** A **smooth manifold** $M$ consists of "patches" (open subsets of some $\mathbb{R}^n$) smoothly knit together.

(Think of smooth surfaces in $\mathbb{R}^3$, like a sphere or torus.)

Each point $p \in M$ has a **tangent space** $T_pM$ attached.

A **Riemannian metric** on $M$ is a smoothly varying collection

$$g = \{g_p : T_pM \times T_pM \to \mathbb{R} \mid p \in M\}$$

of inner products.

A **symplectic form** on $M$ is a smoothly varying collection

$$\omega = \{\omega_p : T_pM \times T_pM \to \mathbb{R} \mid p \in M\}$$

of symplectic products, *such that* $d\omega = 0$*.*

*Every manifold has a Riemannian metric* (partition of unity)*, but not every manifold admits a symplectic form!*

Being even-dimensional and orientable is necessary but not sufficient!

**Example:** $M =$ orientable surface in $\mathbb{R}^3$, $\omega(\vec{u}, \vec{v}) =$ oriented area of parallelogram spanned $\vec{u}$ and $\vec{v}$.

**Fact:** Locally, every symplectic manifold looks like $(\mathbb{R}^{2n}, \omega_0)$. (Darboux's theorem)

(No local invariants in symplectic geometry, like curvature.)

Can define gradients just like before.

$\forall p \in M, \vec{v} \in T_pM, \quad \boxed{df_p(\vec{v}) = \omega_p\left(\nabla_\omega f(p), \vec{v}\right)}$

| | | |
|---|---|---|
| differentiable function | $\rightsquigarrow$ | tangent vector field |
| $f\colon M \to \mathbb{R}$ | $\rightsquigarrow$ | $\nabla f,\ \nabla_\omega f$ |

**Example:** $M = S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, $g =$ dot product, $\omega =$ oriented area

$f\colon S^2 \to \mathbb{R}$, $(x, y, z) \mapsto z$, (height function)

$\nabla f$ points longitudinally, $\nabla_\omega f$ points latitudinally

As before:

$\nabla f$ points to **increasing** values of $f$,
$\nabla_\omega f$ points to **constant** values of $f$.

We can still play "**Find the Hamiltonian!**". Given a tangent vector field $X$ on $M$, can we find a function $f\colon M \to \mathbb{R}$ such that

- $\nabla f = X$?
- $\nabla_\omega f = X$?

# 3. Hamiltonian group actions

**Definition:** A **Lie group** is a group $G$ with a compatible structure of a smooth manifold.

A **smooth action** of $G$ on a smooth manifold $M$ is a "smooth" group homomorphism $\mathcal{A}\colon G \to \text{Diff}(M)$.

$\text{Diff}(M) =$ diffeomorphisms $M \to M$.

The **Lie algebra** $\mathfrak{g}$ of $G$ is the tangent space at the identity element 1 of $G$.

$$\mathfrak{g} := T_1 G$$

$\mathfrak{g}$ is a vector space, and more. (Lie bracket)

**Example:** Some Lie groups.

(i) $(V, +)$.
Lie algebra $\cong V$.

(ii) $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication.
Lie algebra $= i\mathbb{R}$.

(iii) $T = S^1 \times \ldots \times S^1$, a **torus**.
Lie algebra $= i\mathbb{R} \oplus \ldots \oplus i\mathbb{R}$.

(iv) Matrix Lie groups under matrix multiplication, such as $\mathrm{GL}(n;\mathbb{R})$, $\mathrm{SL}(n;\mathbb{R})$, $\mathrm{O}(n;\mathbb{R})$, $\mathrm{SO}(n;\mathbb{R})$, $\mathrm{U}(n)$, etc.
Their Lie algebras are certain matrix vector spaces.

**Example:** A smooth group action. (Rotating the plane.)

Let $G = S^1$, $\mathfrak{g} = i\mathbb{R}$, $M = \mathbb{R}^2$, and $\mathcal{A}\colon S^1 \to$ Diff$(\mathbb{R}^2)$ be defined by

$$\mathcal{A}(e^{i\theta}) \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$1 \in G$ acts as the identity map $M \to M$.

**infinitesimal change** in $G$ at $1 \rightsquigarrow$ **infinitesimal change** at each $p \in M$.

An infinitesimal change at $1 \in G$ is some $\xi \in \mathfrak{g}$.
An infinitesimal change at each $p \in M$ is a vector field.

$$\mathfrak{g} \to \mathrm{Vec}(M), \quad \xi \mapsto \xi_M$$

$\xi_M$ is the **fundamental vector field** on $M$ induced by $\xi$.

$$\xi_M(p) := \frac{d}{dt}\mathcal{A}\left(\exp(t\xi)\right) p \bigg|_{t=0}$$

In the example of rotating the plane, if $\xi = it \in i\mathbb{R} = \mathfrak{g}$, then

$$\xi_M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -ty \\ tx \end{pmatrix}.$$

We can play "**Find the Hamiltonian!**" with the fundamental vector fields.

If we can win this game for every $\xi_M$, then we can form $\phi\colon \mathfrak{g} \to C^\infty(M)$ such that for every $\xi \in \mathfrak{g}$

$$\boxed{\nabla_\omega\,[\phi(\xi)] = \xi_M.}$$

($C^\infty(M) = \{$smooth functions $M \to \mathbb{R}\}$.)

Take "dual", and define $\Phi\colon M \to \mathfrak{g}^*$ by

$$\Phi(p)\xi = \phi(\xi)(p)$$

for all $p \in M$, $\xi \in \mathfrak{g}$.

If $\Phi$ is also $G$**–equivariant** then $\Phi$ is a **moment map** for $\mathcal{A}\colon G \to \mathrm{Diff}(M)$.

$\mathcal{A}$ is a **Hamiltonian action** of $G$ on $M$ if there is a moment map $\Phi$ for the action.

$$\boxed{G \curvearrowright M \xrightarrow{\Phi} \mathfrak{g}^*\,.}$$

**Examples:**

(i) Rotating the plane. $G = S^1$, $\mathfrak{g} = i\mathbb{R}$, $M = \mathbb{R}^2$.
$\Phi\colon M \to \mathfrak{g}^*$ is

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} (it) = \left( \frac{1}{2}(x^2 + y^2) \right) t$$

Note that $\nabla_\omega$ of this function on $\mathbb{R}^2$ is $\begin{pmatrix} -ty \\ tx \end{pmatrix} = (it)_M \begin{pmatrix} x \\ y \end{pmatrix}$.

(ii) $M = \mathbb{R}^6$ with coordinates $\vec{x}, \vec{y} \in \mathbb{R}^3$. ($\vec{x}$ is position, $\vec{y}$ is momentum).

$G = \mathbb{R}^3$ acting on $M$ by translating the position vector. Then $\mathfrak{g} = \mathbb{R}^3 \cong \mathfrak{g}^*$, and $\Phi\colon M \to \mathfrak{g}^*$ is

$$\Phi(\vec{x}, \vec{y})\vec{a} = \vec{y} \cdot \vec{a}.$$

$\Phi =$ **linear momentum**.

(iii) $M =$ cotangent bundle of $\mathbb{R}^3$ with coordinates $\vec{x}, \vec{y} \in \mathbb{R}^3$. ($\vec{x}$ is still position, $\vec{y}$ is still momentum).

$G = SO(3)$ acting on $M$ by "rotation". Then $\mathfrak{g}^* \cong \mathbb{R}^3$, and $\Phi\colon M \to \mathfrak{g}^*$ is

$$\Phi(\vec{x}, \vec{y})\vec{a} = (\vec{x} \times \vec{y}) \cdot \vec{a}.$$

$\Phi =$ **angular momentum**.


EEN

# 4. Atiyah/Guillemin–Sternberg Theorem

Proved independently by Sir Michael Atiyah, and Victor Guillemin and Shlomo Sternberg, in 1982.

**G–S** proof: “simple and elegant”

**A** proof: “even a bit more simple and elegant”

**Thereom:**
$(M, \omega) =$ compact and connected symplectic manifold,
$\mathsf{T} =$ a torus,
$\mathcal{A} =$ Hamiltonian action of $T$ on $M$
with moment map $\Phi\colon M \to \mathfrak{t}^*$.

Then $\Phi(M)$ is a **convex polytope** in $\mathfrak{t}^*$, the **convex hull** of $\Phi(M^{\mathsf{T}})$, where

$$M^{\mathsf{T}} := \{p \in M \mid \mathcal{A}(t)p = p \text{ for all } t \in T\}.$$

# 5. Epilogue

**Symplectic interpretation of Schur-Horn theorem:**

Noticed by Bertram Kostant in the early 1970's, then generalized by A/G–S.

- $U(n) = \{A \mid \bar{A}^T = A^{-1}\}$ is a Lie group. Acts on $\mathcal{H}(n)$ by conjugation. ($\mathcal{H}(n) \cong \mathfrak{u}(n)^*$)

- $T$ = diagonal matrices in $U(n)$ is an $n$–torus. Can identify $\mathfrak{t}^*$ with $\mathbb{R}^n$.

- $\mathcal{O}_{\vec{\lambda}}$ = isospectral set for $\vec{\lambda}$ is a symplectic manifold. (coadjoint orbit, Kirillov–Kostant–Souriau form)

- Conjugation preserves eigenvalues, so $T \curvearrowright \mathcal{O}_{\vec{\lambda}}$.

- $f\colon \mathcal{O}_{\vec{\lambda}} \to \mathbb{R}^n$, $f(A) =$ diagonal of $A$, is a moment map.

- $(\mathcal{O}_{\vec{\lambda}})^T$ = diagonal matrices in $\mathcal{O}_{\vec{\lambda}}$ = diagonal matrices with entries $\lambda_1, \ldots, \lambda_n$ in some order.

**A/G–S** theorem $\Longrightarrow f(\mathcal{O}_{\vec{\lambda}})$ is a convex polytope, the convex hull of $f\left((\mathcal{O}_{\vec{\lambda}})^T\right)$.

This is exactly the **S–H** theorem!

Symplectic stuff is cool! But you don't have to take **my** word for it!

Coming Spring 2008:


Tara Holm's NEW epic

**MATH 758: Symplectic Geometry**

# THE END

*Thank you for listening.*