Math 3110 Preliminary Exam Solutions

1. Prove that there exists $x \in \mathbb{R}$ such that $x = \cos x$.

Let $f(x) = x$, $g(x) = \cos x$. Observe that

$$f(0) = 0 < 1 = g(0)$$

and

$$f(\pi/2) = \pi/2 > 0 = g(\pi/2).$$

Hence, by the intersection principle, there exists $x \in [0, \pi/2]$ such that

$$x = f(x) = g(x) = \cos x.$$

2. Calculate

$$\lim_{n\to\infty} \sqrt{n^2+n} - n.$$

Observe that

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n^2+n} - n &= \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n} + 1} = \frac{1}{2}
\end{aligned}$$

3. Find the radius of convergence of each of the following power series:

We will use the ratio test to find the radius of convergence, R, for each of the following power series:

a) $\sum n^3 x^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)^3 x^{n+1}}{(n)^3 x^n} \right| = \lim_{n\to\infty} (1+1/n)^3 |x| = |x|.$$

Hence, $R = 1$.

b) $\sum \frac{2^n}{n!} x^n$

$$\lim_{n\to\infty} \left| \frac{\frac{(2)^{n+1}x^{n+1}}{(n+1)!}}{\frac{(2)^n x^n}{n!}} \right| = \lim_{n\to\infty} \frac{2}{n+1}|x| = 0.$$

Hence, $R = \infty$.

c) $\sum \frac{2^n}{n^2} x^n$

$$\lim_{n\to\infty} \left| \frac{\frac{(2)^{n+1}x^{n+1}}{(n+1)^2}}{\frac{(2)^n x^n}{n^2}} \right| = \lim_{n\to\infty} \frac{2}{(1+1/n)^2}|x| = 2|x|.$$

Hence, $R = 1/2$.

d) $\sum \frac{n^3}{3^n} x^n$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3 x^{n+1}}{3^{n+1}}}{\frac{n^3 x^n}{3^n}} \right| = \lim_{n\to\infty} (1+1/n)^3 \frac{|x|}{3} = \frac{|x|}{3}.$$

Hence, $R = 3$.

4. Prove that the function f defined by

$$f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$. Draw a graph of the function.

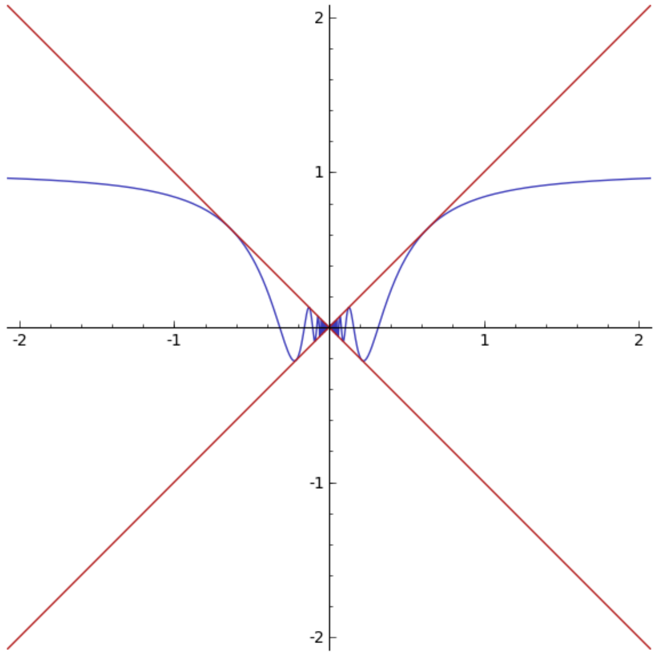
Observe that, since $|\sin\frac{1}{x}| \leq 1$ for all $x \neq 0$,

$$\begin{array}{ccccc} -|x| & \leq x\sin\frac{1}{x} \leq & |x| & & \text{for all } x \neq 0. \\ \downarrow & \downarrow & \downarrow & & \\ 0 & 0 & 0. & & \end{array}$$

Hence, by the squeeze theorem,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0),$$

that is, f is continuous at $x = 0$.



5. Suppose that f is a *continuous* function on $[-1,1]$ such that $x^2 + (f(x))^2 = 1$ for all x. Show that either $f(x) = \sqrt{1-x^2}$ or $f(x) = -\sqrt{1-x^2}$. (Recall that if a is a positive number, then $\sqrt{a}$ denotes the *positive* square root.)

Solving algebraically, we have that,

$$\text{for all } d \in [-1,1], \qquad f(d) = \sqrt{1-d^2} \qquad \text{or} \qquad f(d) = -\sqrt{1-d^2}. \tag{1}$$

From this, we can observe that

$$\text{If } f(d) = 0, \qquad \text{then} \qquad d = 1 \quad \text{or} \quad d = -1. \tag{2}$$

Assume, for the sake of contradiction, that there exists $-1 < a, b < 1$ such that

$$f(a) = \sqrt{1-a^2} \qquad \text{and} \qquad f(b) = -\sqrt{1-b^2}. \tag{3}$$

We are removing the option $a, b = \pm 1$ because $f(1) = f(-1) = 0 = -0$, so the "sign" at $f(1)$ or $f(-1)$ doesn't really matter. Assume that $a \leq b$. (The case $a > b$ is completely analogous.) Then f is continuous in the closed interval $[a, b]$ with $f(a) > 0$ and $f(b) < 0$. By Bolzano's theorem,

$$f(c) = 0 \qquad \text{for some } -1 < a \leq c \leq b < 1.$$

But this contradicts (2). Hence, statement (3) is false and we conclude that the function f satisfies

$$f(x) = \sqrt{1-x^2} \quad \text{for all } x \in [-1,1] \qquad \text{or} \qquad f(x) = -\sqrt{1-x^2} \quad \text{for all } x \in [-1,1]. \tag{4}$$

(Compare statements (1) and (4).)