5.3.4 Suppose $\{a_n\}$ is increasing, $a_n \leq b_n$ for all n, and $b_n \to M$. Prove that a_n converges and $\lim a_n \leq M$.

Proof: As $b_n \to M$, for $\varepsilon = 1$, $\exists N$ s.t. for all $n \geq N$, $|b_n - M| < 1$, $\iff$ $-1 < b_n - M < 1$

In particular this implies that for $n > N$, $b_n < M+1$. As $a_n \leq b_n$ for all n, this implies that for $n \geq N$, $a_n \leq b_n < M+1$. Because $\{a_n\}$ is an increasing sequence, for all $n < N$, we have $a_n \leq a_N \leq b_N < M+1$, so the entire sequence is bounded above by $M+1$.

Because $\{a_n\}$ is an increasing sequence that is bounded above, by the Completeness Property $\{a_n\}$ has a limit.

Now that we know $\{a_n\}$ has a limit, we apply the Limit Location Theorem:

$$a_n \leq b_n \text{ for all } n \Rightarrow \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

It is not true that if $\{a_n\}$ is increasing, $b_n \to M$, and $a_n < b_n$ for all n that $\lim a_n < M$.

Consider $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $\{a_n\}$ is increasing, $b_n \to 0$, and $a_n < b_n$ for all n, but $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$.

5.4.1 Suppose the terms of the sequence $\{a_n\}$ are colored using k different colors, and using each color infinitely often. Then we get k subsequences. If each of the k subsequences converges to L, then $\{a_n\} \to L$ also.

(a) Prove this if $k=2$ and the subsequences are the sequence of odd terms a_{2i+1} and the sequence of even terms a_{2i}.

Let $\{x_n^1\}$ (color #; place in the subsequence) be the sequence of the 1^{st} color, and $\{x_n^2\}$ be the sequence for the 2^{nd}.

* In particular, $x_n^1 := a_{2n}$ for all n and $x_n^2 := a_{2n+1}$ for all n. *

(Notice that we can associate the place in the subsequence (LHS) and the place in the full sequence (RHS))

Given $\varepsilon > 0$.

As $\lim_{n\to\infty} x_n^1 = L$, $\exists N_1$ s.t. for $n > N_1$, $|x_n^1 - L| < \varepsilon$. Let M_1 be the index of $x_{N_1}^1$ in $\{a_n\}$. *$(M_1 = 2N_1)$*

As $\lim_{n\to\infty} x_n^2 = L$, $\exists N_2$ st. for $n > N_2$, $|x_n^2 - L| < \varepsilon$. Let M_2 be the index of $x_{N_2}^2$ in $\{a_n\}$. *$(M_2 = 2N_2+1)$*

Suppose that $n > \max\{M_1, M_2\}$. By hypothesis, $a_n = x_j^1$ or $a_n = x_j^2$ for some j. Without loss of generality, assume $a_n = x_j^1$. Then as $n > \max\{M_1, M_2\} \geq M_1$, we get $j > \max\{N_1, N_2\} \geq N_1$, so $|a_n - L| < \varepsilon$, as desired.

(b) Prove this if $k=2$.

Repeat the proof with segments between *'s deleted.