HW 10 Soln

14.1.2 Let $f(x)=e^x$. Assuming $f'(0)=1$, Prove $f'(x)=e^x$

$$f'(x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x(e^h-e^0)}{h}=e^x\left(\lim_{h\to 0}\frac{e^h-e^0}{h}\right)$$

$$=e^x f'(0)=1.$$

14.1.3 Let $f(x)$ be even and diff'ble at 0. Prove using definitions That $f'(0)=0$. ~~Let x=0.~~

Since $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=L$ exists, $L=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}$.

Since f is even, $f(h)=f(-h)$ so:

$$\overset{L}{\overset{\shortparallel}{f'(0)}}=\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(-h)-f(0)}{h}=-\underbrace{\lim_{h\to 0^+}\frac{f(0)-f(-h)}{h}}_{=\,f'(0)}$$

Hence $L=-L \Rightarrow L=0$, so $\overset{L}{\overset{\shortparallel}{f'(0)}}=0$

//

$\sqrt{1}$

14.1.4 a) Let $f(0)=f'(0)=0$. Find $\lim_{x\to 0} \frac{f(x)}{x}$.

$$0 = f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{x\to 0} \frac{f(x)}{x}.$$

b) Let $|f(x)| \leq x^2$ for $x \approx 0$. Prove $f'(0)=0$.

$\exists\, \delta > 0$ s.t. $|x| \leq \delta \Rightarrow |f(x)| \leq x^2$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{\substack{x\to 0 \\ |x|\leq\delta}} \frac{f(x)-f(0)}{x}$$

Observe $|f(0)| \leq 0^2 \Rightarrow f(0)=0$, so

$f'(0) = \lim_{\substack{x\to 0 \\ |x|\leq\delta}} \frac{f(x)}{x}$. Since $\frac{|f(x)|}{|x|} \leq \frac{x^2}{|x|} = |x|$

for $x \neq 0$, $f'(0)=0$ by the squeeze theorem

because $|x| \to 0$ as $x \to 0$. //

14.1.7 Prove: If $f(x)$ is defined for $x \approx a$ and $\exists$ a number k s.t. $f(x) = f(a) + k(x-a) + e(x)$ where $\lim_{x\to a} \frac{e(x)}{x-a} = 0$

then f is diff'ble at a and $k = f'(a)$.

Pf// $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{f(a)+k(x-a)+e(x)-f(a)}{x-a}$

$$= \underbrace{\lim_{x\to a} \frac{k(x-a)}{x-a} + \lim_{x\to a} \frac{e(x)}{x-a}}_{\text{Both these limits exist, so linearity of limits applies.}} = k \quad //$$

14.2.2 Let $u(x)$ be diff'ble/ non-negative.

Prove $Du^k = k u^{k-1} u'$ if

a) $k = n \in \mathbb{N}^{\geq 1}$

Let $f_n(x) = x^n$. Claim $f_n'(x) = n x^{n-1}$. $*$

For $n=1$: $f_n'(x) = \lim_{h \to 0} \frac{x+h-x}{h} = 1 = 1x^0$.

Assume $f_k'(x) = k x^{k-1}$ $\forall\, k \leq n$.

$f_{n+1}(x) = x^{n+1} = x(x^n)$

If $g(x) = x$, and $h(x) = x^n$, $f_{n+1}'(x) = g'(x)h(x) + g(x)h'(x)$

by the product rule.

Thus by the assumption ~~$f_{n+1}'(x) = (n-1)(x^{n-1})*$~~

$f_{n+1}'(x) = nx^n + x^n = (n+1)x^{n+1}$

and the claim $*$ follows by induction.

The desired result now follows by the chain rule.

b) $k = 1/n$. If $f(x) = x^n$ and $g(x) = x^{1/n}$, on the domain of g,

f and g are inverse functions.

Thus for $x \neq 0$ in the domain of g, $g'(x) = \frac{1}{f'(x)} = [f'(g(x))]^{-1}$

$= \frac{1}{n} (x^{\frac{n-1}{n}})^{-1} = \frac{1}{n} x^{1/n - 1}$

The result now follows by applying the chain rule to $g(u)$.

c. $k = m/n \in \mathbb{Q}$ and $u(x) > 0$.

$u^{m/n} = (u^m)^{1/n}$

$$\left.\begin{aligned} Du^{m/n} &= \frac{1}{n} D(u^m)(u^m)^{1/n-1} \\ &= \frac{m}{n} u^{m-1} (u^m)^{1/n-1} u' \end{aligned}\right\} \text{Chain Rule}$$

$$= \frac{m}{n} u^{m-1+m/n-m} = \frac{m}{n} u^{m/n-1}.$$

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14.2.4 a) If f is diff'ble prove f even $\Rightarrow f'(x)$ odd

$$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0} \frac{+f(-x-h)-f(-x)}{h} = -\lim_{h\to0} \frac{f(-x)-f(-x-h)}{h}$$

$$= -f'(-x)$$

b) " f odd $\Rightarrow f'(x)$ even.

$$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0} \frac{-f(-x-h)+f(-x)}{h} = \lim_{h\to0} \frac{f(-x)-f(-x-h)}{h}$$

$$= f'(-x)$$ //

Note for $t=-h$

$$\lim_{h\to0} \frac{f(-x)-f(-x-h)}{h} = \lim_{t\to0} \frac{f(-x)-f(t-x)}{-t}$$

$$= \lim_{t\to0} \frac{f(-x+t)-f(x)}{t} = f'(-x).$$

14.2.5 If $f(x)$ is diff'ble + periodic, $f'(x)$ is periodic.

Suppose f is periodic with period T,

$$f'(x+T) = \lim_{h\to 0} \frac{f(x+T+h)-f(x+T)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

$\Rightarrow f'$ is periodic with period T

15.1.1 Show how MVT fails (both hypothesis and conclusion) for the following on the interval $(-1,1)$

a) $x^{2/3}$: ~~f~~ at $x=0$ the function is undefined $\Rightarrow$ not cont, not diff'ble on $(-1,1)$

moreover, on $(-1,1)\setminus\{0\}$ $\frac{d\,x^{2/3}}{dx} = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}} \neq 0 \ \forall \begin{pmatrix} x\in(-1,1) \\ x\neq 0\end{pmatrix}$

but $\frac{f(1)-f(-1)}{2} = 0$, so MVT conclusions fail.

b) $f(x) = 1/x$: $f(1)=1$ $f(-1)=(-1)$. ~~as~~ As in part a), ~~where de~~
The function is undefined at $x=0$, so MVT hypotheses fail.

$\frac{f(1)-f(-1)}{2} = 1$, but for $x\in(-1,1)\setminus\{0\}$ $f'(x) = \frac{-1}{x^2} \leq 0 \ \forall x$,

so the MVT conclusion fail's too.

c) $g(x) = \tan(\pi x)$, ~~g~~ ~~g~~ g is undefined at $x = \pm 1/2$, so MVT hypotheses fail.

Also $\frac{g(1)-g(-1)}{2} = 0$, but $g'(x) = \pi\sec^2 \pi x = \frac{\pi}{\cos^2 \pi x} \neq 0$, where g is defined

so MVT conclusions fail.

$\boxed{5}$

15.1.2 Give an example of a function which is not linear, but the conclusions of the MVT hold at infinitely many pts.

for $x \in [0,1]$

e.g. $f(x) = \begin{cases} x \sin \frac{\pi}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$

$f(0) = f(1) = 0$, By elementary calculus $x \sin \pi/x$ is differentiable on $(0,1)$, with derivative $-\frac{\pi}{x} \cos \pi/x + \sin \pi/x = g'(x)$ for $x \in (0,1)$

~~By algebra~~: Also $\lim_{x \to 0^+} x \sin \pi/x = 0$ because $|x \sin \pi/x| \leq |x|$, so $f(x)$ is continuous on $[0,1]$ (using elementary results to get continuity at pts other than 0)

$g'(x) = 0$ if $\tan \frac{\pi}{x} = \frac{\pi}{x}$ (and $g'(x)$ may $= 0$ at other pts too)

Observe that $\frac{\pi}{x} \to \infty$ as $x \to 0$

Since π/x is cont on $(0,1)$, and for all $N \in \mathbb{N}$ $\exists x_N$ s.t. if $0 < x < x_N$ $\pi/x > N$ and $\frac{\pi}{1} = \pi$, by IVT, $\exists$ infinitely many $x \in (0,1)$ s.t. $\frac{\pi}{x} \in \{y + \pi n\}_{n \geq 0, n \in \mathbb{N}}$ for <u>any</u> $y \geq \pi$.

Hence on π/x, every element of the range of tan ~~[illegible]~~ is attained infinitely often by $\tan \pi/x$ because tan is π-periodic.

~~& Hence $\tan \pi/x = \pi/x$ for infinitely many~~

In particular, where $n\pi - \pi/2 < \frac{\pi}{x} < n\pi + \pi/2$ for $n > 1$, $n \in \mathbb{N}$, $\tan \pi/x$ has image $\mathbb{R}$, while π/x is bounded, so $\tan \pi/x$ and π/x must intersect in this interval. This shows $\tan \pi/x$, π/x intersect infinitely often in $(0,1)$, so $g'(x) = 0$ for infinitely many $x \in (0,1)$.

15.1.3 Prove if f is diff'ble on $I=[a,b]$ and changes sign from $-$ to $+$ on I, then $f'(x)>0$ on I for some $x \in I$.

Pf// $\exists\ x_1, x_2 \in I$ s.t. $x_1 < x_2$ and $f(x_1) < 0$, $f(x_2) > 0$.

Hence $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} > 0$.

By MVT, $\exists\ c \in (x_1, x_2) \subseteq I$ s.t. $f'(c) = \dfrac{f(x_2)-f(x_1)}{x_2-x_1} > 0$ //