Homework 6 Solutions

8.1.1. Find the radius of convergence:

(a) $\sum_1 \frac{x^n}{2^n\sqrt{n}}$. Use ratio test.

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{2^{n+1}\sqrt{n+1}}{2^n\sqrt{n}} = \lim_{n\to\infty} 2\sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right)^{1/2} = 2$$

Radius of convergence is 2.

(b) $\sum_1 \frac{(n!)^2}{(2n)!}x^n$ Use ratio test.

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(n!)^2}{(2n)!}\cdot\frac{(2(n+1))!}{((n+1)!)^2} = \lim_{n\to\infty} \frac{(n!)^2}{((n+1)n!)^2}\cdot\frac{(2n+2)!}{(2n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = \lim_{n\to\infty} \frac{(2+2/n)(2+\frac{1}{n})}{(1+1/n)(1+1/n)} = 4$$

Radius of convergence is 4.

(c) $\sum_1 \frac{x^n}{n^{1/n}}$ Use ratio test.

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)^{\frac{1}{n+1}}}{n^{1/n}}$$

Note that the limit of the numerator and denominator are both 1, and are never zero along the way, so $= \frac{\lim_{n\to\infty}(n+1)^{1/n+1}}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1$

Radius of convergence is 1.

(d) $\sum_1 \frac{(-1)^n x^{2n}}{4^n(n!)^2}$ Use ratio test:

$$\lim_{n\to\infty} \frac{|a_{2n+2}x^{2n+2}|}{|a_{2n}x^{2n}|} = \lim_{n\to\infty} \frac{|x|^{2n+2}}{4^{n+1}((n+1)!)^2}\cdot\frac{4^n(n!)^2}{|x|^{2n}} = \lim_{n\to\infty} \frac{|x|^2}{4(n+1)^2} = 0$$

As this is <1 for all x, the series converges for all x. the series has radius of convergence of ∞, ie

(e) $\sum_1 \left(\frac{n+2}{n}\right)^n x^n$ Use root test:

$$\lim_{n\to\infty} (|a_n x^n|)^{1/n} = \lim_{n\to\infty} \left(\left(\frac{n+2}{n}\right)^n |x|^n\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n+2}{n}\right)|x| = |x| < 1 \text{ if } |x|<1.$$

Radius of convergence is 1.

(f) $\sum_{2} \frac{x^n}{(\ln n)^n}$ Use root test

$$\lim_{n\to\infty} (|a_n|^{-1/n}) = \lim_{n\to\infty} \left(\left|\frac{1}{(\ln(n))^n}\right|\right)^{-1/n} = \lim_{n\to\infty} (|\ln(n)|^n)^{1/n} = \lim_{n\to\infty} |\ln(n)| = \infty$$

Radius of convergence is ∞.

(g) $\sum_{1} \frac{n^n x^n}{n!}$ Use ratio test

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n^n}{n!}\frac{(n+1)!}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \cdot \frac{(n+1)!}{(n+1)\cdot n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e}$$

Radius of convergence is $\frac{1}{e}$.

(h) $\sum_{1} \frac{n!\,x^n}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}$ Use ratio test

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)(2n+1)}{1\cdot 3\cdot 5\cdots(2n-1)} = \lim_{n\to\infty} \frac{2n+1}{n+1} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{1+\frac{1}{n}} = 2$$

Radius of convergence is 2.

8.4.1. Let $\sum_0^\infty a_n$ be a series and s_n its sequence of partial sums. Suppose $\sum a_n x^n$ converges for $|x|<1$. Let $f(x) := \sum_0^\infty a_n x^n$. Then $\sum_0^\infty s_n x^n \overset{(*)}{=} \frac{f(x)}{1-x}$ for $|x|<1$.

(b) Prove $(*)$ and show that the hypothesis is satisfied if $\sum a_n$ converges.

Proof: • First, $\sum_0^\infty a_n$ converges means that the series $\sum_0^\infty a_n x^n$ converges for $x=1$. This implies that the radius of convergence for the series is ≥ 1, so in particular, $\sum_0^\infty a_n x^n$ converges for all $|x|<1$.

• Second, notice that for $|x|<1$, we have that $\frac{1}{1-x} = \sum_0^\infty x^n$.

The RHS can be rewritten as a product of series:

$$\frac{f(x)}{1-x} = \underset{\substack{\uparrow \\ \text{Just defin. of } f}}{\left(\sum_0^\infty a_n x^n\right)}\left(\sum_0^\infty x^n\right) = \sum_0^\infty (a_0\cdot 1 + a_1\cdot 1 + \ldots + a_n\cdot 1)x^n = \sum_0^\infty s_n x^n$$

By the product rule for series $(\sum_0^\infty a_n x^n)(\sum_0^\infty b_n x^n) = \sum_0^\infty c_n x^n$ with $c_n = \sum_{i+j=n} a_i b_j$.

Here $b_n = 1$ for every n.

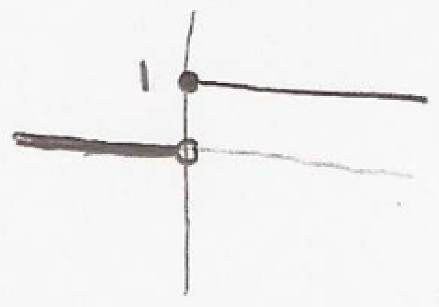
9.2.1 Let $u(x)$ be the unit step function: $u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$

Using algebraic operations, translation, and change of scale, and other composition of functions, express in terms of $u(x)$ the functions

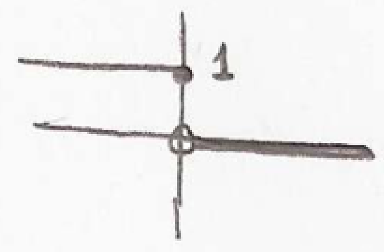
(a) $f(x) = \begin{cases} 2 & 1 \leq x < 3 \\ 0 & \text{elsewhere.} \end{cases}$

There are several solutions. Here's one:

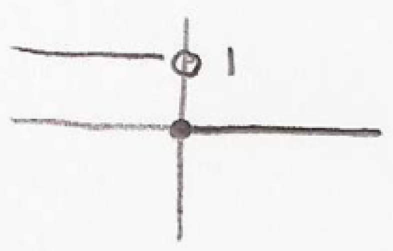
$u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$



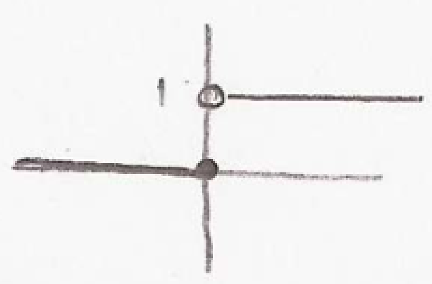
$u(-x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$



$1 - u(x) = \begin{cases} 1 \text{ if } x < 0 \\ 0 \text{ if } x \geq 0 \end{cases}$

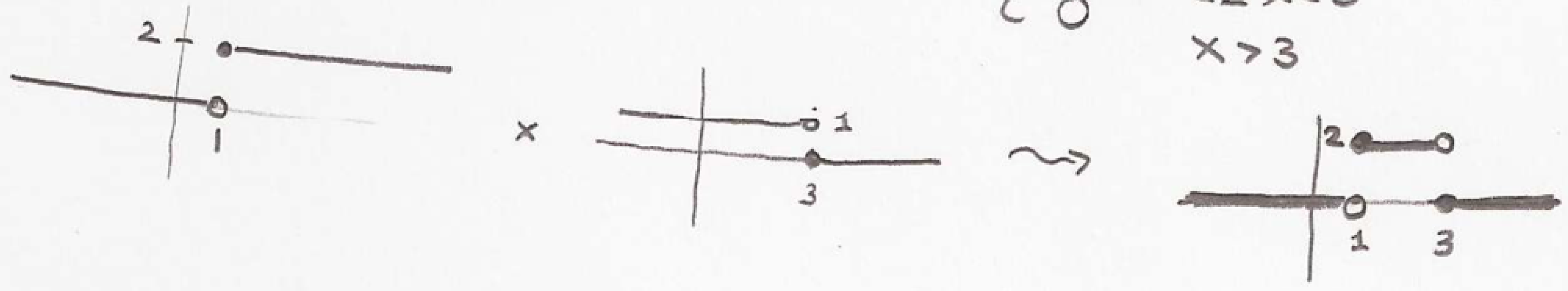


$1 - u(-x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}$

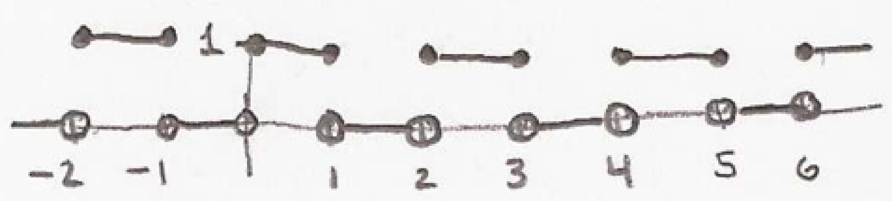


Translate these around to make functions which are non-zero at appropriate places.

$$f(x) = 2u(x-1)(1-u(x-3)) = \begin{cases} 0 & x < 1 \\ 2 & 1 \leq x < 3 \\ 0 & x > 3 \end{cases}$$



$f(x) = \begin{cases} 1 & 2n \leq x \leq 2n+1,\ n \in \mathbb{Z} \\ 0 & \text{elsewhere} \end{cases}$ The graph of this function is



One strategy is to build functions of the form $f_n(x) = \begin{cases} 1 & 2n \leq x \leq 2n+1,\ n \in \mathbb{Z} \\ 0 & \text{elsewhere.} \end{cases}$

and then try to show that $\sum_{n \in \mathbb{Z}} f_n(x)$ converges.

$f_n(x) = u(x-2n)u(2n+1-x)$ for all $n \in \mathbb{Z}$.

Notice that if $x \in [2n, 2n+1]$, then $f_k(x) \neq 0$ iff $k = n$, and if $x \in (2n-1, 2n)$ for some n then $f_k(x) = 0$ for all $k \in \mathbb{Z}$. Therefore $\sum_{n \in \mathbb{Z}} f_n(x) = \begin{cases} 1 & x \in [2n, 2n+1] \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

and this only ever has at most one non-zero term.

9.2.2 Find a function $h(x)$ with this property: for any $f(x)$, the composite function $h(f(x))$ has a graph which agrees with the graph of $f(x)$ where $f(x) \geq 0$, but where $f(x) < 0$, it wipes away that part of the graph of f, and replaces it with the corresponding piece of the x-axis.

$$h(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} = \frac{x + |x|}{2}$$

3. Describe all functions $f(x)$ which are defined for $x \in \mathbb{R}$ and which satisfy $f(x) = \frac{1}{f(x)}$.

If $f(x) = \frac{1}{f(x)}$ for all x, then $f(x)^2 = 1 \Leftrightarrow f(x) \in \{1, -1\}$ for all x.

For $S \subseteq \mathbb{R}$, $f_S(x) = \begin{cases} 1 & x \text{ in } S \\ -1 & x \text{ not in } S \end{cases}$ All functions are of this form for some $S \subseteq \mathbb{R}$.

9.3.1. Suppose $f(x)$ is defined for all x (it would be enough to assume it has a domain D symmetric about 0).

a. Prove that $E(x) = \frac{f(x) + f(-x)}{2}$ is an even function.

To prove even/odd, examine the function at $-x$:

$$E(-x) = \frac{f(-x) + f(--x)}{2} = \frac{f(-x) + f(x)}{2} = E(x) \text{ so } E \text{ is even.}$$

b. Show $f(x)$ can be expressed as the sum of $E(x)$, and an odd function $O(x)$.

$$O(x) := f(x) - E(x) = f(x) - \left(\frac{f(x)}{2} + \frac{f(-x)}{2}\right) = \frac{f(x) - f(-x)}{2}$$

$$O(-x) = \frac{f(-x) - f(--x)}{2} = \frac{f(-x) - f(x)}{2} = (-1)\left(\frac{f(x) - f(-x)}{2}\right) = -O(x). \text{ So } O(x) \text{ is odd and}$$

$f(x)$ is the sum of an even and odd function.

c. This representation is unique. If $f(x) = E_1(x) + O_1(x)$ for E_1 even and O_1 odd, then

$$f(x) = E_1(x) + O_1(x)$$
$$f(-x) = E_1(-x) + O_1(-x) = E_1(x) - O_1(x) \quad \text{as } E_1 \text{ even and } O_1 \text{ is odd}$$

Adding these and dividing by 2, we get

$$E(x) = \frac{f(x) + f(-x)}{2} = \frac{E_1(x) + O_1(x) + E_1(x) - O_1(x)}{2} = \frac{2E_1(x)}{2} = E_1(x).$$

Also $O(x) = f(x) - E(x) = f(x) - E_1(x) = O_1(x)$, so the representation is unique.

d. If $f(x)$ is polynomial, $E(x)$ is the sum of all even degree terms
$O(x)$ is the sum of all odd degree terms.

Indeed, if $f(x) = a_0 + a_1x^1 + \ldots + a_{2k}x^{2k}$ for some k (allowing $a_{2k} = 0$), then by definition,

$$E(x) = \frac{f(x) + f(-x)}{2} = \frac{\sum_{i=0}^{2k} a_i x^i + \sum_{i=0}^{2k} a_i(-x)^i}{2} \quad \text{If } i \text{ is even, } (-x)^i = x^i. \text{ If } i \text{ is odd, } (-x)^i = -x^i.$$

$$= \sum_{i=0}^{k} a_{2i} x^{2i}$$

e. If $f(x) = e^x$, $E(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)$, $O(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$. If you know a power series for e^x, what is the power series for $E(x)$?

2. If f and g are defined for all x and are odd or even, what can be said about the composition $f \circ g$?

Case 1: g even

used g even here

$f \circ g(-x) = f(g(-x)) = f(g(x)) = f \circ g(x)$.

$f \circ g$ is even

Case 2: f even g odd

$f \circ g(-x) = f(g(-x)) \underset{g \text{ odd}}{=} f(-g(x)) \underset{f \text{ even}}{=} f(g(x)) = f \circ g(x)$

$f \circ g$ is even

Case 3: f odd g odd

$f \circ g(-x) = f(g(-x)) = f(-g(x)) = -f(g(x))$

$f \circ g$ is odd

3. Suppose f and g are defined for all x and are decreasing functions. What about $f \circ g$?

Claim: $f \circ g$ is increasing. You can get here by trying an example or two.

If $x \leq y$ then $g(x) \geq g(y)$, as g is decreasing.

As $g(y) \leq g(x)$ then $f(g(y)) \geq f(g(x))$, as f is decreasing.

Therefore, if $x \leq y$ then $f \circ g(x) \leq f \circ g(y)$ ie $f \circ g$ is increasing.

4. Give an example of a function which is periodic, non-constant, and yet has no minimal period.

Consider $f(x) = \begin{cases} 0 & x \text{ is rational} \\ 1 & x \text{ is irrational} \end{cases}$ If c is rational, then $x+c$ is rational iff x is rational, so $c>0$ are all periods of f. In contrast, if c is irrational, $x+c$ is not rational if x was, so c can't be a period of f.

Therefore the set of periods of f are $\{c \mid c>0, c \text{ rational}\}$. This set has infimum of zero, and 0 is not a period, so there's no minimal period. From the def. It is clear that f is non-constant.

5. Show: a periodic increasing function f is constant.

Let $x<y$. We'll show that $f(x)=f(y)$. Therefore f is also decreasing $(x \leq y \Rightarrow f(x) \leq f(y))$ and the only functions which are both increasing and decreasing are constant.

Let c be a period of f. Then take $n > \frac{y-x}{c}$, $n \in \mathbb{N}$.

$f(x) \leq f(y) \leq f(x+nc)$ as $x<y = x + c \cdot \frac{y-x}{c} \leq x + c \cdot n$, and f is increasing.

Because f is periodic and c is one of the periods, $f(x) = f(x+nc)$, so $f(y) = f(x)$.