Hw 8 Solutions

12.1.2 a. Find intervals of unit length where $f(x)=2x^4-8x^3+24x-17$ has its zeros.

Soln: Test ~~unit intervals~~ values $[f(n), f(n+1)]$ for small n to detect sign changes. This yields intervals

$$[-2,-1],\ [0,1],\ [1,2],\ [2,3]$$

12.1.4 a Show that if $0 \le a \le 1/e$, then $xe^{-x}=a$ has a non-negative solution.

Observe if $f(x)=xe^{-x}$, f is a product of elementary functions and is hence continuous.

Then $f(0)=0$ and $f(1)=\frac{1}{e}$, $f(0)\le a \le f(1)$, so by the IVT

$\exists\ 0\le c\le 1$ s.t. $f(c)=a$.

b) If $0<a<1/e$ there are two non-negative soln to $f(x)=a$.

From calculus, $\lim_{x\to\infty} f(x)=0$. Thus $\exists\ N\in\mathbb{N}$ s.t if $x\ge N$,

$|f(x)|<a$. Hence since $f(1)>a>f(N)$, by IVT, $\exists$

$d\in(1,N)$ s.t. $f(d)=a$.

Then c,d are distinct non-negative solns.

12.1.5. Show that for values of $y \geq 1$, $y^2 \cos x - e^x = 0$ has a soln for ~~$x \in [0, \pi/2]$~~ $x \in [0, \pi/2)$.

Observe if $f(x) = y^2 \cos x - e^x$, $f(0) = y^2 - 1 \geq 0$

while $f(\pi/2) = -e^{\pi/2} \leq 0$

~~In fact, if π~~

Hence by IVT ~~f is continuous as a product~~ (because f is formed by algebraic manipulation of elementary functions, it is cont), $\exists\ c \in [0, \pi/2]$ s.t. $f(c) = 0$.

Now $f(\pi/2) = e^{-\pi/2} \neq 0$, so $c \in [0, \pi/2)$.

17.2.1 Assuming reasonably that the height function is continuous, the problem is, show a continuous function on a circle has the property that $\exists$ a pair of diametrically opposite pts which have the same evaluation.

Let h be the height function on our circle.

We can view h as $h(\theta): \mathbb{R} \to \mathbb{R}$, a 2π periodic function where θ is the ~~angle~~ unique pt on the unit circle in $\mathbb{R}^2$ at angle θ in polar coordinates.

Let $g(\theta) = h(\theta) - h(\theta + \pi)$. Then observe $g(\theta) = -g(\theta + \pi)$ b/c $h(\theta + 2\pi) = h(\theta)$. If $g(0) = 0$, we are done.

Otherwise by IVT, $\exists\ \theta \in [0, \pi]$ s.t. $g(\theta) = 0$, so when $g(\theta) = 0$, diametrically opposite pts on the diameter at angle θ have the same value i.e. $h(\theta) = h(\theta + \pi)$.

12.4.1) Show $\exists$ exactly one real fifth root of a number $a \in \mathbb{R}$.

Note: The function $\sqrt[5]{x}$ is defined only because the result of this problem holds. Therefore, you <u>may not</u> use it here.

Recall, $\lim_{x \to \infty} x^5 = \infty$ and $\lim_{x \to -\infty} x^5 = -\infty$.

Thus $\exists$ $N \in \mathbb{N}$ s.t. ~~[illegible]~~ if $x \geq N$, $x^5 > a$.

" if $x \leq -N$ $x^5 < a$

Hence ~~[illegible]~~ if $f(x) = x^5$, $f(-N) \leq a \leq f(N)$, so by

IVT, $\exists$ c s.t. $-N \leq c \leq N$ s.t. $c^5 = f(c) = a$.

Hence c is a 5^{th} root of a.

Observe that $f(x)$ is strictly increasing, so c must be the unique real 5^{th} root of a.

Note: a actually has up to $\underline{5}$ distinct 5^{th} roots, but the remaining roots ~~[illegible]~~ are complex.

e.g. The 5^{th} roots of 1 are

~~$\sin(2\pi/5) + i\cos$~~

1, $\cos(2\pi/5) + i\sin(2\pi/5)$, $\cos(4\pi/5) + i\sin(4\pi/5)$

$\cos(6\pi/5) + i\sin(6\pi/5)$, $\cos(8\pi/5) + i\sin(8\pi/5)$

12-2 Let $f(x)$ be cont on $I = [a,b] \subseteq \mathbb{R}$ and let $f(I) = [f(a), f(b)]$

Suppose $f(x)$ is injective i.e. $f(x) = f(y) \Rightarrow x = y$.

Prove $f(x)$ is strictly increasing.

Suppose not, Then $\exists$ $x_1, x_2 \in [a,b]$ s.t. $f(x_1) \geq f(x_2)$

but $x_1 \neq x_2$, $x_1 < x_2$. In fact, by f injective, $f(x_1) > f(x_2)$

Observe if $x_1 = a$, $x_2 = b$, Then we have a contradiction.

Thus $x_1 \neq a$ or $x_2 \neq b$. Assume the former, The other case is similar.

If $f(x_2) > f(a)$, Then by IVT, $\exists$ $a < c < x_1$ s.t. $f(x_2) = f(c)$

$\Rightarrow \Leftarrow$ b/c f injective.

If $f(x_2) < f(a)$, then by IVT, $\exists$ $x_1 < d < x_2$ s.t. $f(a) = f(d)$

$\Rightarrow \Leftarrow$ b/c f injective.

12-3 Let f be a cont fcn on $[-a, a]$. Suppose $f(0) > f(-a)$, $f(0) > f(a)$.

Show That There is a chord of length a.

Let $g(x) = f(x+a) - f(a)$, so $g: [-a, 0] \to \mathbb{R}$.

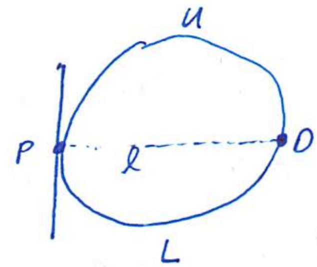
Observe $g(-a) = f(0) - f(a) > 0$ but

$g(0) = f(a) - f(0) < 0$

so by Bolzano thm g has a root ~~between~~ on $[-a, 0]$ at c.

Thus There is a chord between $(c, f(c))$ and $(c+a, f(c+a))$.

12-5 Let C be a smooth convex closed curve.
Show you can always inscribe an equilateral triangle in C.



Fix a point P on C.

~~Let Q be the point in C s.t.~~

Let l be a perpendicular to the tangent of C at P, and let D be the other point where l intersects C (D is unique by convexity).

Partition C into halves U, L using l.

~~Choose a pt~~ The choice of a pt Q on U, continuously determines a segment PQ.

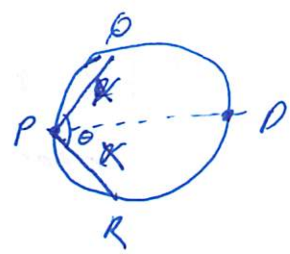
Using IVT, there exists a unique pt. R on L s.t.

length(PQ) = length(PR) s.t. R is the closest such pt (in distance on C)

~~Now specifying an angle θ for $\theta = PQR$ continuously determines Q and R~~

If we specify x, there is a continuous fcn $f: \mathbb{R} \to \mathbb{R}$ s.t. if Q is the point on U w/ length$(PQ) = x$ and R is the pt on L s.t length$(PR) = x$,

$\theta = \angle PQR$,



set $f(x) = \theta$

for $x = 0$, $f(0) = 0$

and for $x = $ length(PD), $f(x) = \pi$,

So $\exists\, x$ s.t. $f(x) = \pi/3$ where $0 \le x \le$ length(PD) by IVT.

The triangle PQR where PQ, PR have the same length and $\angle PQR = \pi/3$ is equilateral.

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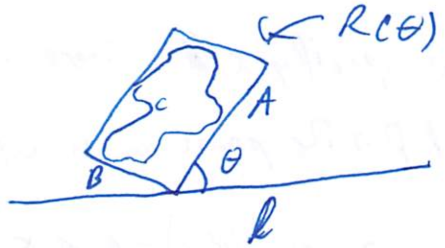
12-6. Show that if C is a continuous closed curve, C can be ~~inscribed in a~~ circumscribed by a square.

C is closed, so C can be inscribed in a minimal rectangle with side labelled A perpendicular to the x-axis in the plane

~~The segment A continuously~~

Let $R(\theta) : [0, \pi/2] \to$ Minimal Rectangles enclosing C be a function that determines the minimal rectangle such that θ is the counterclockwise angle made by a rectangle with a horizontal which intersects R, but ~~contains no points in~~ no points of R lie in the half plane below it

i.e. $R(\theta)$ is the minimal rectangle containing C s.t.



Label A as the side which makes the angle θ with a horizontal line, and label B as a side of opposite length

Set $d(\theta) = \text{length}(A) - \text{length}(B)$, which is determined continuously.

By minimality $R(0), R(\pi/2)$ are the same rectangle but A, B switch places, so $d(\theta)$ changes sign, if $d(0) \neq 0$.

hence $d(\theta) = 0$ for some $\theta \in [0, \pi/2]$

At that θ, $R(\theta)$ is a square.

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12.3.1 Show that a ~~continuous~~ function f with the IVP on $[a,b]$ which is strictly ~~continuous~~ decreasing is continuous.

Pf// Let $x_0 \in [a,b]$ and $\varepsilon > 0$ be given.

If ~~f(x)~~ $\exists x$ such that $f(x_0) - f(x) > \varepsilon$

Then $x > x_0$ b/c f decr. hence $\exists x_0 > x_2 > x$ s.t.

$f(x_0) > f(x_2) > \cancel{f(x_0)-\varepsilon}\ f(x) > f(x_0) - \varepsilon$ by IVP

Similarly if $\exists x$ s.t. ~~$f(x_0)$~~ $f(x) - f(x_0) > \varepsilon$,

Then $x < x_0$ and $\exists x < x_1 < x_0$ s.t. $f(x_0) + \varepsilon > f(x_1) > f(x_0)$.

~~If x_1 was defined~~ If we defined neither x_1 nor x_2,

Then let $\delta > 0$.

If we defined x_1 but not x_2, set $\delta = |x_1 - x_0|$

" x_2 " x_1, set $\delta = |x_2 - x_0|$

Otherwise set $\delta = \min(|x_1 - x_0|, |x_2 - x_0|)$.

Then $\forall x \in [a,b]$ s.t. $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$

b/c f is strictly decreasing.

Thus f is cont on $[a,b]$.